

Multivariable Analysis - Homework 2

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Your grade is computed as: $10n/13$, where n is the number of correctly answered subquestions.

1. Let V be a 4-dimensional vector space.

(a) Choose a basis \mathbf{b} for V and write down explicit bases for $\Lambda^i V$, $0 \leq i \leq 4$, in terms of \mathbf{b} .

Solution: Let (b_1, b_2, b_3, b_4) be the basis for V , that is $\mathbf{b}(e_i) = b_i$. By Lemma 1.7.1, a basis for $\Lambda^i V$ is given by $\bigwedge_{i \in I} b_i$ where I is an increasing sequence of length i whose members are $\{1, 2, 3, 4\}$. This means that

$$\text{Basis for } \Lambda^1 V : \quad (b_1, b_2, b_3, b_4)$$

$$\text{Basis for } \Lambda^2 V : \quad (b_1 \wedge b_2, b_1 \wedge b_3, b_1 \wedge b_4, b_2 \wedge b_3, b_2 \wedge b_4, b_3 \wedge b_4)$$

$$\text{Basis for } \Lambda^3 V : \quad (b_1 \wedge b_2 \wedge b_3, b_1 \wedge b_2 \wedge b_4, b_1 \wedge b_3 \wedge b_4, b_2 \wedge b_3 \wedge b_4)$$

$$\text{Basis for } \Lambda^4 V : \quad (b_1 \wedge b_2 \wedge b_3 \wedge b_4)$$

On the other hand, since $\Lambda^0 V := \text{span}(\mathbf{1})$, then $\mathbf{1}$ is its only base element.

(b) Explain why property (6) of the definition of exterior algebra implies that $b_1 \wedge b_2$ is linearly independent from $b_1 \wedge b_2 \wedge b_4$.

Solution: Suppose $\lambda b_1 \wedge b_2 + \mu b_1 \wedge b_2 \wedge b_4 = 0$. Since $0 \in \Lambda V$ admits a decomposition as $0 = 0 + 0 + \dots + 0$, where the i -th summand is the zero of $\Lambda^i V$, then $\lambda b_1 \wedge b_2 = 0$ and $\mu b_1 \wedge b_2 \wedge b_4 = 0$. Since $b_1 \wedge b_2$ and $b_1 \wedge b_2 \wedge b_4$ form part of some basis by (a), they are nonzero and hence $\lambda = \mu = 0$.

(c) Let $y = \mathbf{1} + b_1 + b_1 \wedge b_2 + b_1 \wedge b_2 \wedge b_3 + b_1 \wedge b_2 \wedge b_3 \wedge b_4$. Find the unique elements $y_i \in \Lambda V$ such that $y \wedge y = y_0 + y_1 + y_2 + y_3 + y_4$.

Solution: We compute

$$\begin{aligned} y \wedge y &= [(y - \mathbf{1}) + \mathbf{1}] \wedge [(y - \mathbf{1}) + \mathbf{1}] \\ &= (y - \mathbf{1}) \wedge (y - \mathbf{1}) + 2(y - \mathbf{1}) + \mathbf{1} \\ &= \mathbf{1} + 2b_1 + 2b_1 \wedge b_2 + 2b_1 \wedge b_2 \wedge b_3 + 2b_1 \wedge b_2 \wedge b_3 \wedge b_4 \end{aligned}$$

where the last equality follows because in the product $(y - \mathbf{1}) \wedge (y - \mathbf{1})$ every summand contains two copies of b_1 , and therefore it is zero. Thus $y_0 = \mathbf{1}$, $y_1 = 2b_1$, $y_2 = 2b_1 \wedge b_2$, $y_3 = 2b_1 \wedge b_2 \wedge b_3$ and $y_4 = 2b_1 \wedge b_2 \wedge b_3 \wedge b_4$.

- (d) Find an element $x \in \Lambda^2 V$ such that $x \wedge x \neq 0$.

Solution: Let $x := b_1 \wedge b_2 + b_3 \wedge b_4$. Then

$$\begin{aligned} x \wedge x &= (b_1 \wedge b_2) \wedge (b_1 \wedge b_2) + (b_1 \wedge b_2) \wedge (b_3 \wedge b_4) + \\ &\quad (b_3 \wedge b_4) \wedge (b_1 \wedge b_2) + (b_3 \wedge b_4) \wedge (b_3 \wedge b_4) \\ &= b_1 \wedge b_2 \wedge b_3 \wedge b_4 + b_3 \wedge b_4 \wedge b_1 \wedge b_2 \\ &= 2b_1 \wedge b_2 \wedge b_3 \wedge b_4 \neq 0. \end{aligned}$$

- (e) For any $\phi \in V^*$ compute the induced map $\Lambda^2 \phi : \Lambda^2 V \rightarrow \Lambda^2 \mathbb{R}$.

Solution: In first place observe that $\Lambda^2 \mathbb{R} = 0$, because if e_1 is a basis for \mathbb{R} , by Lemma 1.7.1, $e_1 \wedge e_1 = 0$ is a basis for $\Lambda^2 \mathbb{R}$. But then $\Lambda^2 \phi$ has to be the zero map, because it is the only map with target the trivial vector space.

2. In this exercise we investigate the function $f : P \rightarrow \mathbb{R}^{n+1}$ where $P = \{x \in \mathbb{R}^n : |x|^2 < 4\}$ and $f(p) = (p, \sqrt{4 - |p|^2})$. As usual we take the Euclidean norm on \mathbb{R}^n which is the square root of the sum of the squares of the coefficients with respect to the standard basis.

- (a) Set $n = 1$ and compute $f'(p)$ using property 2) and 5) of Theorem 2.1.1. Please keep in mind that for fixed p your final answer $f'(p)$ is supposed to be a linear map.

Solution: By property 5, f is differentiable in P (because its coordinate functions are) and the linear map $f'(p) : \mathbb{R} \rightarrow \mathbb{R}^2$ is represented (in the standard basis) by the matrix whose entries are the derivatives of the coordinate functions, that is

$$f'(p) = \begin{pmatrix} 1 \\ \frac{-p}{\sqrt{4-p^2}} \end{pmatrix}$$

where we have used property 2 in the second entry (which in this case took the form of the 1-variable chain rule).

- (b) Keeping $n = 1$ compute $T_{f(p)}(f)$. Is this set tangent to the image $f(P)$?

Solution: By definition $T_{f(p)}(f) = \text{Im } f'(p) = \text{span}(e_1 - \frac{p}{\sqrt{4-p^2}}e_2)$. If eg $p = 0$, we see that $T_{f(p)}(f) = \text{span}(e_1)$ which is indeed tangent to the semicircle $f(P)$ (up to a translation, because linear subspaces must contain the origin).

- (c) From now on n is any positive integer. Introduce $J : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ by $J(x) = (x, x)$. Verify that J is a linear map and compute J'

Solution: J is linear because $J(\lambda x + y) = (\lambda x + y, \lambda x + y) = (\lambda x, \lambda x) + (y, y) = \lambda(x, x) + (y, y) = \lambda J(x) + J(y)$. By Theorem 2.1.1.(4) we automatically have $J'(p) = J$ for all $p \in P$.

- (d) Show that $f = f^1 + e_{n+1}(f^2)$ where $f^1 \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^{n+1})$ is determined by $f^1(e_i) = e_i$ for $i = 1 \dots n$ and $f^2 = u \circ N$ where $u : (-4, 4) \rightarrow \mathbb{R}$ is defined by $u(x) = \sqrt{4-x}$ and $N : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as $N = g \circ J$ where g is the Euclidean inner product.

Solution: Given an element $x \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, write $x = (x', x_{n+1})$ where $x' \in \mathbb{R}^n$ contains its first n coordinates and x_{n+1} is its last coordinate. Then we see that f^1 is the map $f^1(p) = (p, 0)$. On the other hand

$$(u \circ g \circ J)(p) = (u \circ g)(p, p) = u(|p|^2) = \sqrt{4 - |p|^2} = f^2(p).$$

Putting all together we see that

$$(f^1 + e_{n+1}f^2)(p) = f^1(p) + e_{n+1}f^2(p) = (p, 0) + (0, \sqrt{4 - |p|^2}) = (p, \sqrt{4 - |p|^2}).$$

- (e) Explain from the definition of the derivative why $J'(p) = J$ in the sense that for all $p, v \in \mathbb{R}^n$ we have $J'(p)(v) = J(v)$. Also argue that for the same reason $(f^1)'(p) = f^1$.

Solution: Recall that the derivative $g'(p)$ of a map $g : V \rightarrow W$ is the unique linear map satisfying $\lim_{v \rightarrow 0} \frac{g(p+v) - g(p) - g'(p)v}{|v|} = 0$. Since

$$\lim_{v \rightarrow 0} \frac{J(p+v) - J(p) - J(v)}{|v|} = \lim_{v \rightarrow 0} \frac{J(p) + J(v) - J(p) - J(v)}{|v|} = 0,$$

we conclude that $J'(p) = J$ for all $p \in P$, by the uniqueness of the derivative. A similar argument works for f^1 :

$$\lim_{v \rightarrow 0} \frac{f^1(p+v) - f^1(p) - f^1(v)}{|v|} = \lim_{v \rightarrow 0} \frac{(p+v, 0) - (p, 0) - (v, 0)}{|v|} = 0,$$

and we conclude that $(f^1)'(p) = f^1$ for all $p \in P$, by the uniqueness of the derivative.

- (f) Use the chain rule to prove $N'(p) = g'(J(p)) \circ J$.

Solution: By the chain rule, $N'(p) = (g \circ J)'(p) = g'(J(p)) \circ J'(p) = g'(J(p)) \circ J$, where the last equality follows from (e).

- (g) Use the chain rule once more to show $(f^2)'(p)(v) = 2u'(|p|^2)g(p, v)$. Why did the composition in the chain rule become product of real numbers here?

Solution: Since $f^2 = u \circ N$, we simply compute

$$\begin{aligned} (f^2)'(p)v &= (u \circ N)'(p)v \\ &= (u'(N(p)) \circ N'(p))v \\ &= (u'(|p|^2) \circ g'(J(p)) \circ J)v \\ &= (u'(|p|^2) \circ g'(p, p))(v, v) \\ &= 2u'(|p|^2)g(p, v), \end{aligned}$$

where in the third equality we used (f) and in the fourth one the calculation of $g'(p, q)(v, w)$ from page 32. The chain rule became a product of real numbers because the last function to be composed, u , is a 1-variable function.

- (h) Finally write an expression for $f'(p)$ and describe $T_{f(p)}(f)$.

Solution: By property 5 of the derivative, (e) and (g) we have

$$\begin{aligned} f'(p)v &= (f^1 + e_{n+1}(f^2))'(p)v \\ &= (f^1)'(p)v + (f^2)'(p)v \\ &= f^1(v) - \frac{g(p, v)}{\sqrt{4 - |p|^2}}e_{n+1} \\ &= v_1e_1 + \cdots + v_n e_n - \frac{g(p, v)}{\sqrt{4 - |p|^2}}e_{n+1}, \end{aligned}$$

where v_i is the i -th coordinate of v . We conclude computing the tangent space $T_{f(p)}(f)$:

$$\begin{aligned} T_{f(p)}(f) &= \text{Im } f'(p) \\ &= \text{span} \{f'(p)e_1, \dots, f'(p)e_n\} \\ &= \text{span} \left\{ e_1 - \frac{g(p, e_1)}{\sqrt{4 - |p|^2}}e_{n+1}, \dots, e_n - \frac{g(p, e_n)}{\sqrt{4 - |p|^2}}e_{n+1} \right\} \\ &= \text{span} \left\{ e_1 - \frac{p_1}{\sqrt{4 - |p|^2}}e_{n+1}, \dots, e_n - \frac{p_n}{\sqrt{4 - |p|^2}}e_{n+1} \right\}, \end{aligned}$$

where p_i is the i -th coordinate of p .