

# Multivariable Analysis - Homework 1

Jorge Becerra, Oscar Koster and Laurens Pieper-Espada

November 24, 2020

Your grade is computed as:  $10 * n/7$  where  $n$  is the number of correctly answered subquestions.

1. (a) Find a linear isomorphism  $L \in \text{Hom}(A, B)$  from  $A = \text{span}(e_1 + e_2 + e_3, e_2 + e_3) \subset \mathbb{R}^3$  to  $B = \ker f$  where  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  is defined by  $f(a, b, c) = -a + b + c$ .

**Solution:**

First notice that  $A$  can be rewritten to be  $\text{span}(e_1, e_2 + e_3)$  because  $e_1 = e_1 + e_2 + e_3 - (e_2 + e_3)$ . Similarly, we can rewrite the kernel of  $f$  as the set  $\{a, b, c \in \mathbb{R} : a = b + c\}$ . So we can write any vector in  $B$  as:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b + c \\ b \\ c \end{pmatrix} = b(e_1 + e_2) + c(e_1 + e_3).$$

Therefore,  $B = \text{span}(e_1 + e_2, e_1 + e_3)$ .

We can now define a map  $L : A \rightarrow B$ , by defining  $L$  on the basis vectors of  $A$ . Let  $L(e_1) = e_1 + e_3$  and  $L(e_2 + e_3) = e_1 + e_3$ .

Note that  $L$  is a linear map and because both  $A$  and  $B$  are two-dimensional vector spaces it has an inverse  $L^{-1}$ , which can be defined by taking the basis vectors of  $B$  to the basis vectors of  $A$ . Therefore, we have given an isomorphism  $L$ .

- (b) Suppose  $S \subset V$  is some set of linearly independent vectors. If  $F, G \in \text{Hom}(V, W)$  are such that  $F(v) = G(v)$  for all  $v \in S$ , is it true that  $F = G$ ?

**Solution:** This is not true in general. A counterexample is given by  $F, G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $F(e_1) = e_1$  and  $F(e_2) = e_2$  and  $G(e_1) = e_1$  and  $G(e_2) = 2e_2$ . Moreover, let  $S = \{e_1\}$  be a subset of  $\mathbb{R}^2$ . Then note that  $F(v) = G(v)$  for all vectors  $v$  in  $S$ , but  $F$  and  $G$  are different functions.

- (c) Is it true that if  $F(e_i) = G(e_i)$  for all  $i = 1, \dots, n$  and  $F, G \in \text{Hom}(\mathbb{R}^n, W)$  then  $F = G$ ?

**Solution:** This is true because any linear map is fully determined by image of the basis vectors. So given the image of the basis vectors is the same, the linear maps must be the same.

2. Consider  $V = \mathbb{R}^3$  and  $L \in \text{Hom}(V, V)$  defined by  $L(e_1) = \frac{3}{2}e_1 + e_2$ ,  $L(e_2) = \frac{1}{4}e_1 + \frac{3}{2}e_2$  and  $L(e_3) = -\frac{1}{4}e_1 - \frac{1}{2}e_2 + e_3$ .

(a) Verify that  $\mathbf{c} \in \text{Hom}(\mathbb{R}^3, V)$  given by  $\mathbf{c}(e_1) = e_1 + 2e_2$ ,  $\mathbf{c}(e_2) = e_2 + e_3$  and  $\mathbf{c}(e_3) = e_1 + e_2 + 3e_3$  is a basis for  $V$ .

**Solution:** We need to check  $\mathbf{c} : \mathbb{R}^3 \rightarrow V$  is an isomorphism. Because  $\mathbf{c}$  is a linear map, we only need to check that it is invertible. Notice, that the matrix associated to this linear map is given by:

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

This matrix has determinant equal to 4, and hence the matrix is non-singular and invertible. This proves  $\mathbf{c}$  is an invertible linear map, and hence a linear isomorphism.

(b) Give two other examples of bases  $\mathbf{a}, \mathbf{b} : \mathbb{R}^3 \rightarrow V$ .

**Solution:** Any two linear maps from  $\mathbb{R}^3$  to  $V$  with an associated non-singular  $3 \times 3$  matrix with respect to the standard basis would suffice for this question. However, note that not all these matrices are easy to compute and choosing a different basis might make some problems significantly less complicated.

(c) Write down the matrices for  $\mathbf{a}^{-1}L\mathbf{a}$ ,  $\mathbf{b}^{-1}L\mathbf{b}$  and  $\mathbf{c}^{-1}L\mathbf{c}$ .

**Solution:** Work out the matrix multiplication with your chosen matrices. You could do this by hand, however since this is not a course about computation it is fine to use Mathematica or Matlab or any other nice software to do this computation.

The matrix  $\mathbf{c}^{-1}L\mathbf{c}$  is given by:

$$\mathbf{c}^{-1}L\mathbf{c} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) Also write down the matrices for  $\mathbf{a}^{-1}L\mathbf{b}$  and  $\mathbf{c}^{-1}L\mathbf{a}$ .

**Solution:** See question (c).