Additional exercises for "Algebraic Topology"

27th September 2018

1. Let S be a set, and A an abelian group. The *A-linearization* of S was defined as

$$A[S] := \{f : S \longrightarrow A : f^{-1}(A - 0) \text{ is finite }\}.$$

For $a \in A$ and $s \in S$, denote as the map sending s to a and everything else to 0. Then every element $f \in A[S]$ can be expressed in a unique way as $f = a_1 s_1 + \cdots + a_n s_n$, where $a_i \in A$ and $s_n \in S$.

Now let $(A_i)_{i \in I}$ be a collection of abelian groups. The *direct sum* $\bigoplus_{i \in I} A_i$ is the collection of tuples $(a_i)_{i \in I}$ where only finite many a_i are nonzero. It has a structure of abelian group by addition termwise.

(a) Show that $\mathbb{Z}[S]$ is isomorphic to $\bigoplus_{s \in S} \mathbb{Z}$. The latter is usually called the *free abelian group generated by S* .

(b) Observe that there is a natural map of sets $i : S \longrightarrow \mathbb{Z}[S]$ sending $s \in S$ to $1 \cdot s \in \mathbb{Z}[S]$. Show that the free abelian group has the following property: if A is an abelian group and $\varphi : S \longrightarrow A$ is a map of sets, there is a unique group homomorphism $\widetilde{\varphi} : \mathbb{Z}[S] \longrightarrow A$ such that $\widetilde{\varphi} \circ i = \varphi$.

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "i"'] & A \\ \mathbb{Z}[S] \arrow[ur, dashed, "\widetilde{\varphi}"'] & \end{tikzcd}$$

In other words,

$$\mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}[S], A) = \mathrm{Hom}_{\mathsf{Sets}}(S, A).$$

(c) Show that the previous property is *universal*: if $F(S)$ is another abelian group together with a map $j : S \longrightarrow F(S)$ such that the latter property is fulfilled, then there exists a unique group isomorphism $\psi : \mathbb{Z}[S] \longrightarrow F(S)$ such that $i \circ \psi = j$.

$$\begin{tikzcd} & \mathbb{Z}[S] \arrow[dd, dashed, "\psi"] \\ S \arrow[ur, "i"] \arrow[dr, "j"'] & \\ & F(S) \end{tikzcd}$$

Hint: Use (a) twice; or use the Yonneda lemma if you know it.

(d) Show that the construction of the free abelian group is *functorial*, that is, if S, T are sets and $f : S \longrightarrow T$ is a map of sets, there is a unique group homomorphism $\mathbb{Z}[f] : \mathbb{Z}[S] \longrightarrow \mathbb{Z}[T]$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "i"'] & T \arrow[d, "j"] \\ \mathbb{Z}[S] \arrow[r, dashed, "\mathbb{Z}[f]"] & \mathbb{Z}[T] \end{tikzcd}$$

Hint: Use (a).

2. Let $\{E_n, F_n : n \geq 0\}$ be a collection of vector spaces, and set $C_n := E_n \oplus F_n \oplus E_{n-1}$.

(a) Show that projection-inclusion maps $\partial_n : C_n \longrightarrow E_{n-1} \hookrightarrow C_{n-1}$ make C into a chain complex.

(b) Show that every chain complex of vector spaces is isomorphic to a chain complex of this form.

Hint. For a chain complex of vector spaces (C, ∂), set $E_n := \operatorname{Im} \partial_{n+1}$ and $F_n := H_{n+1}(C)$.

3. Given groups G and H, the set $\operatorname{Hom}(G, H)$ of group homomorphisms $f : G \longrightarrow H$ is again a group homomorphism, by setting $(f + f')(g) := f(g) + f'(g)$ and with unit element the zero map.

   Let (C, ∂) be a chain complex of abelian groups, and let A be an abelian group. Show that $\{\operatorname{Hom}(A, C_n)\}$ forms a chain complex of abelian groups.

4. A chain complex C is called *acyclic* if $H_n(C) = 0$ for all $n > 0$. Give an example of such a chain complex.

5. A chain map $f : C \longrightarrow D$ is called a *quasi-isomorphism* if it induces isomorphisms $f_* : H_n(C) \longrightarrow H_n(D)$ in all homology groups.

   Show that for a chain complex C the following are equivalent:

   (a) C is an exact chain complex.

   (b) C is acyclic.

   (c) The zero map $0 \longrightarrow C$ is a quasi-isomorphism, where 0 denotes the chain complex given by trivial groups and trivial differentials.

6. Let $f : C \longrightarrow D$ be a chain map. The *mapping cone* of f is the chain complex $\operatorname{Cone} f$ defined by

$$(\operatorname{Cone} f)_n := D_n \oplus C_{n-1} \qquad , \qquad \partial_n^{\operatorname{Cone} f}(x, y) := (\partial_n^D x + f_{n-1} y, -\partial_{n-1} y).$$

   (a) Check that $(\operatorname{Cone} f, \partial^{\operatorname{Cone} f})$ is indeed a chain complex.

   (b) For a chain complex C, its *shifted* chain complex $C[1]$ is given by

$$C[1]_n := C_{n-1} \qquad , \qquad \partial_n^{C[1]} = -\partial_{n-1}.$$

   Show that $H_n(C[1]) = H_{n-1}(C)$.

   (c) Show that the canonical injection and projection induce a short exact sequence of chain complexes

$$0 \longrightarrow D \longrightarrow \operatorname{Cone} f \longrightarrow C[1] \longrightarrow 0$$

7. Let $\{C^i\}_{i \in I}$ be a family of chain complexes indexed by a set I.

   (a) Show that setting

$$(\bigoplus_i C^i)_n = \bigoplus_i C^i_n \qquad , \qquad \partial(a_i)_{i \in I} = (\partial a_i)_{i \in I}$$

   defines a chain complex $\bigoplus_i C^i$.

   (b) Show that the canonical injections $\iota_{C^i_n} : C^i_n \hookrightarrow \bigoplus_i C^i_n$ induce a chain map $\iota_{C^i} : C^i \longrightarrow \bigoplus_i C^i$.

   (c) Show that $\bigoplus_i C^i$ has the following universal property: given a chain complex D and a family of chain maps $f_i : C^i \longrightarrow D$, there is a unique chain map $f : \bigoplus_i C^i \longrightarrow D$ such that $f \circ \iota_{C^i} = f$. In other words,

$$\operatorname{Hom}_{\mathrm{Ch}}(\bigoplus_i C^i, D) = \prod_i \operatorname{Hom}_{\mathrm{Ch}}(C^i, D).$$

   (d) Show that there is an isomorphism

$$\bigoplus_i H_n(C^i) \longrightarrow H_n(\bigoplus C^i)$$

   whose composite with $\iota_{H_n(C^i)}$ is the map $H_n(C^i) \longrightarrow H_n(\bigoplus_i C^i)$ induced by the chain map ι_{C^i}.

8. The aim of the following exercises is to recall some notions of point-set topology which will appear along the course.

   (a) Let X be a topological space. Its *suspension* SX is the quotient of $X \times I$ by the equivalence relation $(x,0) \sim (y,0)$ for all $x, y \in X$ and $(x,1) \sim (y,1)$ for all $x, y \in X$.
   Show that $S\mathbb{S}^n$ is homeomorphic to $\mathbb{S}^{n+1}$.

   (b) The *real projective space* $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1} - \{0\}$ by the equivalence relation $x \sim y \iff y = \lambda x$ for some $\lambda \in \mathbb{R} - \{0\}$.
   Show that $\mathbb{RP}^n$ is homeomorphic to the quotient of $\mathbb{S}^n$ by the equivalence relation which identifies antipodal points, $x \sim -x$.

   (c) Let $I = [0,1]$ The *torus* $\mathbb{T}$ is the quotient of I^2 by the equivalence relation which identifies $(t,0) \sim (t,1)$ for all $t \in I$ and $(0,s) \sim (1,s)$ for all $t \in I$.
   Show that the $\mathbb{T}$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$.

   Hint: Use the *universal property of the quotient topology*: let $f : X \longrightarrow Y$ be a continuous map, let $\sim$ be a equivalent relation on X and let $\pi : X \longrightarrow X/\sim$ be the canonical projection. Then there exists a continuous map $\bar{f} : X/\sim \longrightarrow Y$ such that $\bar{f} \circ \pi = f$ if and only if f satisfies that whenever $x \sim y$, then $f(x) = f(y)$.

   You might also want to use that a bijective, continuous map with compact source and Hausdorff target is a homeomorphism.

9. Give an example of a pair of spaces (X, X') such that $H_n(X'; A)$ is not a subgroup of $H_n(X; A)$ for some $n \in \mathbb{N}$ (formally, that the inclusion $X' \hookrightarrow X$ does not induce an injection in homology).

   Give an example of a pair of spaces (X, X') such that $H_n(X'; A)$ is a subgroup of $H_n(X; A)$ for some $n \in \mathbb{N}$ and the quotient $H_n(X; A)/H_n(X'; A)$ is isomorphic to $H_n(X, X'; A)$.

10. Let (X, X') be a pair of spaces, let A be an abelian group and let $x \in X'$. Show that if $H_n(X', \{x\}; A) \cong 0$, then the map of pairs $(X, \{x\}) \longrightarrow (X, X')$ induces an isomorphism

$$H_n(X, \{x\}; A) \cong H_n(X, X'; A).$$

11. Let (X, X') be a pair of spaces and let A be an abelian group.

   (a) Show that $H_0(X, X'; A) \cong 0$ if and only if X' meets all path-components of X.

   (b) Show that $H_1(X, X'; A) \cong 0$ if and only if the map $H_1(X'; A) \longrightarrow H_1(X; A)$ induced by the inclusion $X' \hookrightarrow X'$ is surjective and every path-component of X meets at most one path-component of X'.

   Hint: Argue with the long exact sequence of a pair.

12. Let $p_1, \ldots, p_m \in \mathbb{S}^2$ be different points on the sphere. Compute all homology groups $H_n(\mathbb{S}^2, \{p_1, \ldots, p_m\}; A)$ for all $n \geq 0$ and all abelian groups A.