Proof of Proposition 3.10:

In[∘]:=

```
(*We establish the multiplication for SW. We
 set the variable l to be sw. The bar denotes inverse. *)
vars = {s, w, l};
```

$$m_{i\_,j\_\to k\_}[Z\_] := \text{Expand}[Z] /. \left\{s_i\, s_j \to 1,\ w_i\, w_j \to 0,\ s_i\, w_j \to l_k,\ w_i\, s_j \to -l_k,\ l_i\, s_j \to -w_k,\ s_i\, l_j \to w_k,\ l_i\, w_j \to 0,\ w_i\, l_j \to 0,\ l_i\, l_j \to 0,\ x\_{}_i\ /;\ \text{MemberQ}[vars, x] :\to x_k,\ x\_{}_j\ /;\ \text{MemberQ}[vars, x] :\to x_k\right\}$$

In[∘]:=

```
(* We start by declaring the elements R and κ: *)
```

$$\kappa_{i\_} := \lambda_1 + \lambda_2\, s_i + \lambda_3\, w_i + \lambda_4\, l_i;$$

$$\bar{\kappa}_{i\_} := \bar{\lambda}_1 + \bar{\lambda}_2\, s_i + \bar{\lambda}_3\, w_i + \bar{\lambda}_4\, l_i;$$

$$R_{i\_,j\_} := \mu_{1,1} + \mu_{1,2}\, s_i + \mu_{1,3}\, w_i + \mu_{1,4}\, l_i + \mu_{2,1}\, s_j + \mu_{2,2}\, s_i\, s_j + \mu_{2,3}\, w_i\, s_j + \mu_{2,4}\, l_i\, s_j + \mu_{3,1}\, w_j + \mu_{3,2}\, s_i\, w_j + \mu_{3,3}\, w_i\, w_j + \mu_{3,4}\, l_i\, w_j + \mu_{4,1}\, l_j + \mu_{4,2}\, s_i\, l_j + \mu_{4,3}\, w_i\, l_j + \mu_{4,4}\, l_i\, l_j$$

$$\bar{R}_{i\_,j\_} := \bar{\mu}_{1,1} + \bar{\mu}_{1,2}\, s_i + \bar{\mu}_{1,3}\, w_i + \bar{\mu}_{1,4}\, l_i + \bar{\mu}_{2,1}\, s_j + \bar{\mu}_{2,2}\, s_i\, s_j + \bar{\mu}_{2,3}\, w_i\, s_j + \bar{\mu}_{2,4}\, l_i\, s_j + \bar{\mu}_{3,1}\, w_j + \bar{\mu}_{3,2}\, s_i\, w_j + \bar{\mu}_{3,3}\, w_i\, w_j + \bar{\mu}_{3,4}\, l_i\, w_j + \bar{\mu}_{4,1}\, l_j + \bar{\mu}_{4,2}\, s_i\, l_j + \bar{\mu}_{4,3}\, w_i\, l_j + \bar{\mu}_{4,4}\, l_i\, l_j$$

```
(*We also implement the element ν :*)
```

$$\nu_{i\_} := R_{1,3}\, \bar{\kappa}_2\ //\ m_{1,2\to 1}\ //\ m_{1,3\to i}$$

$$\bar{\nu}_{i\_} := \bar{R}_{3,1}\, \bar{\kappa}_2\ //\ m_{1,2\to 1}\ //\ m_{1,3\to i}$$

In[∘]:=

```
(*By the previous lemma, we simply set the values of the bar variables: *)
```

$$\left\{\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4\right\} = \left\{\frac{\lambda_1}{\lambda_1^2 - \lambda_2^2}, -\frac{\lambda_2}{\lambda_1^2 - \lambda_2^2}, -\frac{\lambda_3}{\lambda_1^2 - \lambda_2^2}, -\frac{\lambda_4}{\lambda_1^2 - \lambda_2^2}\right\};$$

rulesRinv =

$$\Big\{\bar{\mu}_{1,1} \to -\Big(\big(-\mu_{1,1}^3 + \mu_{1,1}\mu_{1,2}^2 + \mu_{1,1}\mu_{2,1}^2 - 2\mu_{1,2}\mu_{2,1}\mu_{2,2} + \mu_{1,1}\mu_{2,2}^2\big) \big/ \big(\mu_{1,1}^4 - 2\mu_{1,1}^2\mu_{1,2}^2 + \mu_{1,2}^4 - 2\mu_{1,1}^2\mu_{2,1}^2 - 2\mu_{1,2}^2\mu_{2,1}^2 + \mu_{2,1}^4 + 8\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2} - 2\mu_{1,1}^2\mu_{2,2}^2 - 2\mu_{1,2}^2\mu_{2,2}^2 - 2\mu_{2,1}^2\mu_{2,2}^2 + \mu_{2,2}^4\big)\Big),$$

$$\bar{\mu}_{1,2} \to -\Big(\big(\mu_{1,1}^2\mu_{1,2} - \mu_{1,2}^3 + \mu_{1,2}\mu_{2,1}^2 - 2\mu_{1,1}\mu_{2,1}\mu_{2,2} + \mu_{1,2}\mu_{2,2}^2\big) \big/ \big(\mu_{1,1}^4 - 2\mu_{1,1}^2\mu_{1,2}^2 + \mu_{1,2}^4 - 2\mu_{1,1}^2\mu_{2,1}^2 - 2\mu_{1,2}^2\mu_{2,1}^2 + \mu_{2,1}^4 + 8\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2} - 2\mu_{1,1}^2\mu_{2,2}^2 - 2\mu_{1,2}^2\mu_{2,2}^2 - 2\mu_{2,1}^2\mu_{2,2}^2 + \mu_{2,2}^4\big)\Big),$$

$$\bar{\mu}_{1,3} \to -\Big(\big(\mu_{1,1}^2\mu_{1,3} - \mu_{1,2}^2\mu_{1,3} + \mu_{1,3}\mu_{2,1}^2 - \mu_{1,3}\mu_{2,2}^2 - 2\mu_{1,1}\mu_{2,1}\mu_{2,3} + 2\mu_{1,2}\mu_{2,2}\mu_{2,3}\big) \big/ \big(\mu_{1,1}^4 - 2\mu_{1,1}^2\mu_{1,2}^2 + \mu_{1,2}^4 - 2\mu_{1,1}^2\mu_{2,1}^2 - 2\mu_{1,2}^2\mu_{2,1}^2 + \mu_{2,1}^4 + 8\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2} - 2\mu_{1,1}^2\mu_{2,2}^2 - 2\mu_{1,2}^2\mu_{2,2}^2 - 2\mu_{2,1}^2\mu_{2,2}^2 + \mu_{2,2}^4\big)\Big),$$

$$\bar{\mu}_{1,4} \to -\Big(\big(\mu_{1,1}^2\mu_{1,4} - \mu_{1,2}^2\mu_{1,4} + \mu_{1,4}\mu_{2,1}^2 - \mu_{1,4}\mu_{2,2}^2 - 2\mu_{1,1}\mu_{2,1}\mu_{2,4} + 2\mu_{1,2}\mu_{2,2}\mu_{2,4}\big) \big/ \big(\mu_{1,1}^4 - 2\mu_{1,1}^2\mu_{1,2}^2 + \mu_{1,2}^4 - 2\mu_{1,1}^2\mu_{2,1}^2 - 2\mu_{1,2}^2\mu_{2,1}^2 + \mu_{2,1}^4 + 8\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2} - 2\mu_{1,1}^2\mu_{2,2}^2 - 2\mu_{1,2}^2\mu_{2,2}^2 - 2\mu_{2,1}^2\mu_{2,2}^2 + \mu_{2,2}^4\big)\Big),$$

$$\bar{\mu}_{2,1} \to -\Big(\big(\mu_{1,1}^2\mu_{2,1} + \mu_{1,2}^2\mu_{2,1} - \mu_{2,1}^3 - 2\mu_{1,1}\mu_{1,2}\mu_{2,2} + \mu_{2,1}\mu_{2,2}^2\big) \big/ \big(\mu_{1,1}^4 - 2\mu_{1,1}^2\mu_{1,2}^2 + \mu_{1,2}^4 - 2\mu_{1,1}^2\mu_{2,1}^2 - 2\mu_{1,2}^2\mu_{2,1}^2 + \mu_{2,1}^4 + 8\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2} - 2\mu_{1,1}^2\mu_{2,2}^2 - 2\mu_{1,2}^2\mu_{2,2}^2 - 2\mu_{2,1}^2\mu_{2,2}^2 + \mu_{2,2}^4\big)\Big),$$

$$\bar{\mu}_{2,2} \to -\Big(\big(-2\mu_{1,1}\mu_{1,2}\mu_{2,1} + \mu_{1,1}^2\mu_{2,2} + \mu_{1,2}^2\mu_{2,2} + \mu_{2,1}^2\mu_{2,2} - \mu_{2,2}^3\big) \big/ \big(\mu_{1,1}^4 - 2\mu_{1,1}^2\mu_{1,2}^2 + \mu_{1,2}^4 - 2\mu_{1,1}^2\mu_{2,1}^2 - 2\mu_{1,2}^2\mu_{2,1}^2 + \mu_{2,1}^4 + 8\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2} - 2\mu_{1,1}^2\mu_{2,2}^2 - 2\mu_{1,2}^2\mu_{2,2}^2 - 2\mu_{2,1}^2\mu_{2,2}^2 + \mu_{2,2}^4\big)\Big),$$

$$\bar{\mu}_{2,3} \to -\Big(\big(-2\mu_{1,1}\mu_{1,3}\mu_{2,1} + 2\mu_{1,2}\mu_{1,3}\mu_{2,2} + \mu_{1,1}^2\mu_{2,3} - \mu_{1,2}^2\mu_{2,3} + \mu_{2,1}^2\mu_{2,3} - \mu_{2,2}^2\mu_{2,3}\big) \big/ \big(\mu_{1,1}^4 - 2\mu_{1,1}^2\mu_{1,2}^2 + \mu_{1,2}^4 - 2\mu_{1,1}^2\mu_{2,1}^2 - 2\mu_{1,2}^2\mu_{2,1}^2 + \mu_{2,1}^4 + 8\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2} - 2\mu_{1,1}^2\mu_{2,2}^2 - 2\mu_{1,2}^2\mu_{2,2}^2 - 2\mu_{2,1}^2\mu_{2,2}^2 + \mu_{2,2}^4\big)\Big),$$

$$\bar{\mu}_{2,4} \to -\Big(\big(-2\mu_{1,1}\mu_{1,4}\mu_{2,1} + 2\mu_{1,2}\mu_{1,4}\mu_{2,2} + \mu_{1,1}^2\mu_{2,4} - \mu_{1,2}^2\mu_{2,4} + \mu_{2,1}^2\mu_{2,4} - \mu_{2,2}^2\mu_{2,4}\big) \big/ \big(\mu_{1,1}^4 - 2\mu_{1,1}^2\mu_{1,2}^2 + \mu_{1,2}^4 - 2\mu_{1,1}^2\mu_{2,1}^2 - 2\mu_{1,2}^2\mu_{2,1}^2 + \mu_{2,1}^4 + 8\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2} - 2\mu_{1,1}^2\mu_{2,2}^2 - 2\mu_{1,2}^2\mu_{2,2}^2 - 2\mu_{2,1}^2\mu_{2,2}^2 + \mu_{2,2}^4\big)\Big),$$

$$\bar{\mu}_{3,1} \to -\Big(\big(\mu_{1,1}^2\mu_{3,1} + \mu_{1,2}^2\mu_{3,1} - \mu_{2,1}^2\mu_{3,1} - \mu_{2,2}^2\mu_{3,1} - 2\mu_{1,1}\mu_{1,2}\mu_{3,2} + 2\mu_{2,1}\mu_{2,2}\mu_{3,2}\big) \big/$$

$$\left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\Big),\ \overline{\mu}_{3,2} \to$$

$$-\Big(\left(-2\,\mu_{1,1}\,\mu_{1,2}\,\mu_{3,1} + 2\,\mu_{2,1}\,\mu_{2,2}\,\mu_{3,1} + \mu_{1,1}^2\,\mu_{3,2} + \mu_{1,2}^2\,\mu_{3,2} - \mu_{2,1}^2\,\mu_{3,2} - \mu_{2,2}^2\,\mu_{3,2}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\Big),$$

$$\overline{\mu}_{3,3} \to -\Big(\left(-2\,\mu_{1,1}\,\mu_{1,3}\,\mu_{3,1} + 2\,\mu_{2,1}\,\mu_{2,3}\,\mu_{3,1} + 2\,\mu_{1,2}\,\mu_{1,3}\,\mu_{3,2} - 2\,\mu_{2,2}\,\mu_{2,3}\,\mu_{3,2} + \mu_{1,1}^2\,\mu_{3,3} - \mu_{1,2}^2\,\mu_{3,3} - \mu_{2,1}^2\,\mu_{3,3} + \mu_{2,2}^2\,\mu_{3,3} + 2\,\mu_{1,4}\,\mu_{2,2}\,\mu_{4,1} - 2\,\mu_{1,2}\,\mu_{2,4}\,\mu_{4,1} - 2\,\mu_{1,4}\,\mu_{2,1}\,\mu_{4,2} + 2\,\mu_{1,1}\,\mu_{2,4}\,\mu_{4,2} + 2\,\mu_{1,2}\,\mu_{2,1}\,\mu_{4,4} - 2\,\mu_{1,1}\,\mu_{2,2}\,\mu_{4,4}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\Big),$$

$$\overline{\mu}_{3,4} \to -\Big(\left(-2\,\mu_{1,1}\,\mu_{1,4}\,\mu_{3,1} + 2\,\mu_{2,1}\,\mu_{2,4}\,\mu_{3,1} + 2\,\mu_{1,2}\,\mu_{1,4}\,\mu_{3,2} - 2\,\mu_{2,2}\,\mu_{2,4}\,\mu_{3,2} + \mu_{1,1}^2\,\mu_{3,4} - \mu_{1,2}^2\,\mu_{3,4} - \mu_{2,1}^2\,\mu_{3,4} + \mu_{2,2}^2\,\mu_{3,4} + 2\,\mu_{1,3}\,\mu_{2,2}\,\mu_{4,1} - 2\,\mu_{1,2}\,\mu_{2,3}\,\mu_{4,1} - 2\,\mu_{1,3}\,\mu_{2,1}\,\mu_{4,2} + 2\,\mu_{1,1}\,\mu_{2,3}\,\mu_{4,2} + 2\,\mu_{1,2}\,\mu_{2,1}\,\mu_{4,3} - 2\,\mu_{1,1}\,\mu_{2,2}\,\mu_{4,3}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\Big),$$

$$\overline{\mu}_{4,1} \to -\Big(\left(\mu_{1,1}^2\,\mu_{4,1} + \mu_{1,2}^2\,\mu_{4,1} - \mu_{2,1}^2\,\mu_{4,1} - \mu_{2,2}^2\,\mu_{4,1} - 2\,\mu_{1,1}\,\mu_{1,2}\,\mu_{4,2} + 2\,\mu_{2,1}\,\mu_{2,2}\,\mu_{4,2}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\Big),\ \overline{\mu}_{4,2} \to$$

$$-\Big(\left(-2\,\mu_{1,1}\,\mu_{1,2}\,\mu_{4,1} + 2\,\mu_{2,1}\,\mu_{2,2}\,\mu_{4,1} + \mu_{1,1}^2\,\mu_{4,2} + \mu_{1,2}^2\,\mu_{4,2} - \mu_{2,1}^2\,\mu_{4,2} - \mu_{2,2}^2\,\mu_{4,2}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\Big),$$

$$\overline{\mu}_{4,3} \to -\Big(\left(2\,\mu_{1,4}\,\mu_{2,2}\,\mu_{3,1} - 2\,\mu_{1,2}\,\mu_{2,4}\,\mu_{3,1} - 2\,\mu_{1,4}\,\mu_{2,1}\,\mu_{3,2} + 2\,\mu_{1,1}\,\mu_{2,4}\,\mu_{3,2} + 2\,\mu_{1,2}\,\mu_{2,1}\,\mu_{3,4} - 2\,\mu_{1,1}\,\mu_{2,2}\,\mu_{3,4} - 2\,\mu_{1,1}\,\mu_{1,3}\,\mu_{4,1} + 2\,\mu_{2,1}\,\mu_{2,3}\,\mu_{4,1} + 2\,\mu_{1,2}\,\mu_{1,3}\,\mu_{4,2} - 2\,\mu_{2,2}\,\mu_{2,3}\,\mu_{4,2} + \mu_{1,1}^2\,\mu_{4,3} - \mu_{1,2}^2\,\mu_{4,3} - \mu_{2,1}^2\,\mu_{4,3} + \mu_{2,2}^2\,\mu_{4,3}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\Big),$$

$$\overline{\mu}_{4,4} \to -\Big(\left(2\,\mu_{1,3}\,\mu_{2,2}\,\mu_{3,1} - 2\,\mu_{1,2}\,\mu_{2,3}\,\mu_{3,1} - 2\,\mu_{1,3}\,\mu_{2,1}\,\mu_{3,2} + 2\,\mu_{1,1}\,\mu_{2,3}\,\mu_{3,2} + 2\,\mu_{1,2}\,\mu_{2,1}\,\mu_{3,3} - 2\,\mu_{1,1}\,\mu_{2,2}\,\mu_{3,3} - 2\,\mu_{1,1}\,\mu_{1,4}\,\mu_{4,1} + 2\,\mu_{2,1}\,\mu_{2,4}\,\mu_{4,1} + 2\,\mu_{1,2}\,\mu_{1,4}\,\mu_{4,2} - 2\,\mu_{2,2}\,\mu_{2,4}\,\mu_{4,2} + \mu_{1,1}^2\,\mu_{4,4} - \mu_{1,2}^2\,\mu_{4,4} - \mu_{2,1}^2\,\mu_{4,4} + \mu_{2,2}^2\,\mu_{4,4}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\Big)\Big\};$$

```
Scan[Set @@ # &, rulesRinv]
```

In[ ]:=

```
(*Sanity check:*)
```

$\kappa_1\,\overline{\kappa}_2$ // $\mathrm{m}_{1,2\to1}$ // Simplify

$\kappa_1\,\overline{\kappa}_2$ // $\mathrm{m}_{2,1\to1}$ // Simplify

$\mathrm{R}_{1,2}\,\overline{\mathrm{R}}_{3,4}$ // $\mathrm{m}_{1,3\to1}$ // $\mathrm{m}_{2,4\to1}$ // Simplify

$\mathrm{R}_{1,2}\,\overline{\mathrm{R}}_{3,4}$ // $\mathrm{m}_{3,1\to1}$ // $\mathrm{m}_{4,2\to1}$ // Simplify

Out[ ]=

1

Out[ ]=

1

Out[ ]=

1

Out[ ]=

1

(*Let us now analyse the commutators. We start with $[\mathrm{R}, \kappa\otimes 1]$:*)

In[ ]:=
```
vari = {s_1, w_1, l_1, s_2, w_2, l_2};
variL = {"s_1", "w_1", "l_1", "s_2", "w_2", "l_2"};
p = (R_{1,2} κ_3 // m_{1,3→1}) - (R_{1,2} κ_3 // m_{3,1→1});
cr = CoefficientRules[p, vari];
convert[{v___Integer}] := StringJoin@Pick[variL, {v}, 1];
(convert[First[#]] → Last[#]) & /@ cr
```

Out[ ]=

$$\{w_1 s_2 \to 2\,\lambda_4\,\mu_{2,2} - 2\,\lambda_2\,\mu_{2,4},\ w_1 w_2 \to 2\,\lambda_4\,\mu_{3,2} - 2\,\lambda_2\,\mu_{3,4},\ w_1 l_2 \to 2\,\lambda_4\,\mu_{4,2} - 2\,\lambda_2\,\mu_{4,4},\ w_1 \to 2\,\lambda_4\,\mu_{1,2} - 2\,\lambda_2\,\mu_{1,4},$$
$$l_1 s_2 \to 2\,\lambda_3\,\mu_{2,2} - 2\,\lambda_2\,\mu_{2,3},\ l_1 w_2 \to 2\,\lambda_3\,\mu_{3,2} - 2\,\lambda_2\,\mu_{3,3},\ l_1 l_2 \to 2\,\lambda_3\,\mu_{4,2} - 2\,\lambda_2\,\mu_{4,3},\ l_1 \to 2\,\lambda_3\,\mu_{1,2} - 2\,\lambda_2\,\mu_{1,3}\}$$

```
(*We have indicated the monomial to which each coefficient corresponds
 to.  All one has to show is that the coefficients not in J⊗J are zero,
for instance "2 γ_4 μ_{2,2}-2 γ_2 μ_{2,4} =0" for w_1s_2. The strategy of
  the proof is to show that these must be consequences of the XC-
 axioms. We will show that these are consequences of only XC0c. For this we
  will compute a Groebner basis for the coefficients of the relation XC0c.*)
```

In[ ]:=
```
vari = {s_1, w_1, l_1, s_2, w_2, l_2};
variL = {"s_1", "w_1", "l_1", "s_2", "w_2", "l_2"};
p = ((κ_1 κ_2 R_{3,4} κ̄_5 κ̄_6 // m_{1,3→1} // m_{1,5→1} // m_{2,4→2} // m_{2,6→2}) - R_{1,2});
cr = CoefficientRules[p, vari] // Simplify // Expand;
convert[{v___Integer}] := StringJoin@Pick[variL, {v}, 1];
(convert[First[#]] → Last[#]) & /@ cr
eqs = Thread[Values[cr]] // Simplify // Expand;
```

Out[ ]=

$$\Big\{s_1 s_2 \to 0,\ s_1 w_2 \to -\frac{2\,\lambda_2\,\lambda_3\,\mu_{2,2}}{\lambda_1^2-\lambda_2^2} - \frac{2\,\lambda_1\,\lambda_4\,\mu_{2,2}}{\lambda_1^2-\lambda_2^2} + \frac{2\,\lambda_2^2\,\mu_{3,2}}{\lambda_1^2-\lambda_2^2} + \frac{2\,\lambda_1\,\lambda_2\,\mu_{4,2}}{\lambda_1^2-\lambda_2^2},$$

$$s_1 l_2 \to -\frac{2\,\lambda_1\,\lambda_3\,\mu_{2,2}}{\lambda_1^2-\lambda_2^2} - \frac{2\,\lambda_2\,\lambda_4\,\mu_{2,2}}{\lambda_1^2-\lambda_2^2} + \frac{2\,\lambda_1\,\lambda_2\,\mu_{3,2}}{\lambda_1^2-\lambda_2^2} + \frac{2\,\lambda_2^2\,\mu_{4,2}}{\lambda_1^2-\lambda_2^2},$$

$$s_1 \to 0,\ w_1 s_2 \to -\frac{2\,\lambda_2\,\lambda_3\,\mu_{2,2}}{\lambda_1^2-\lambda_2^2} - \frac{2\,\lambda_1\,\lambda_4\,\mu_{2,2}}{\lambda_1^2-\lambda_2^2} + \frac{2\,\lambda_2^2\,\mu_{2,3}}{\lambda_1^2-\lambda_2^2} + \frac{2\,\lambda_1\,\lambda_2\,\mu_{2,4}}{\lambda_1^2-\lambda_2^2},$$

$$w_1 w_2 \to \frac{4\,\lambda_2^2\,\lambda_3^2\,\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{8\,\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4\,\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{4\,\lambda_1^2\,\lambda_4^2\,\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_1^2\,\lambda_2\,\lambda_3\,\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_2^3\,\lambda_3\,\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_1^3\,\lambda_4\,\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_1\,\lambda_2^2\,\lambda_4\,\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2} -$$

$$\frac{4\,\lambda_1\,\lambda_2^2\,\lambda_3\,\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{4\,\lambda_1^2\,\lambda_2\,\lambda_4\,\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_1^2\,\lambda_2\,\lambda_3\,\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_2^3\,\lambda_3\,\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_1^3\,\lambda_4\,\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_1\,\lambda_2^2\,\lambda_4\,\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{4\,\lambda_1^2\,\lambda_2^2\,\mu_{3,3}}{(\lambda_1^2-\lambda_2^2)^2} +$$

$$\frac{2\,\lambda_1^3\,\lambda_2\,\mu_{3,4}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{2\,\lambda_1\,\lambda_2^3\,\mu_{3,4}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{4\,\lambda_1\,\lambda_2^2\,\lambda_3\,\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{4\,\lambda_1^2\,\lambda_2\,\lambda_4\,\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{2\,\lambda_1^3\,\lambda_2\,\mu_{4,3}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{2\,\lambda_1\,\lambda_2^3\,\mu_{4,3}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{4\,\lambda_1^2\,\lambda_2^2\,\mu_{4,4}}{(\lambda_1^2-\lambda_2^2)^2},$$

$$w_1 l_2 \to \frac{4\,\lambda_1\,\lambda_2\,\lambda_3^2\,\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{4\,\lambda_1^2\,\lambda_3\,\lambda_4\,\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{4\,\lambda_2^2\,\lambda_3\,\lambda_4\,\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{4\,\lambda_1\,\lambda_2\,\lambda_4^2\,\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_1^3\,\lambda_3\,\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_1\,\lambda_2^2\,\lambda_3\,\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2} -$$

$$\frac{2\,\lambda_1^2\,\lambda_2\,\lambda_4\,\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_2^3\,\lambda_4\,\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{4\,\lambda_1^2\,\lambda_2\,\lambda_3\,\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{4\,\lambda_1\,\lambda_2^2\,\lambda_4\,\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{4\,\lambda_1\,\lambda_2^2\,\lambda_3\,\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{4\,\lambda_1^2\,\lambda_2\,\lambda_4\,\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2} +$$

$$\frac{2\,\lambda_1^3\,\lambda_2\,\mu_{3,3}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{2\,\lambda_1\,\lambda_2^3\,\mu_{3,3}}{(\lambda_1^2-\lambda_2^2)^2} + \frac{4\,\lambda_1^2\,\lambda_2^2\,\mu_{3,4}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_1^2\,\lambda_2\,\lambda_3\,\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_2^3\,\lambda_3\,\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_1^3\,\lambda_4\,\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2} - \frac{2\,\lambda_1\,\lambda_2^2\,\lambda_4\,\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2} +$$

$$\frac{4\lambda_1^2\lambda_2^2\mu_{4,3}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{2\lambda_1^3\lambda_2\mu_{4,4}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{2\lambda_1\lambda_2^3\mu_{4,4}}{(\lambda_1^2-\lambda_2^2)^2},\ \mathtt{w}_1\to-\frac{2\lambda_2\lambda_3\mu_{1,2}}{\lambda_1^2-\lambda_2^2}-\frac{2\lambda_1\lambda_4\mu_{1,2}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_2^2\mu_{1,3}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_1\lambda_2\mu_{1,4}}{\lambda_1^2-\lambda_2^2},$$

$$\mathtt{l}_1\mathtt{s}_2\to-\frac{2\lambda_1\lambda_3\mu_{2,2}}{\lambda_1^2-\lambda_2^2}-\frac{2\lambda_2\lambda_4\mu_{2,2}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_1\lambda_2\mu_{2,3}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_2^2\mu_{2,4}}{\lambda_1^2-\lambda_2^2},$$

$$\mathtt{l}_1\mathtt{w}_2\to\frac{4\lambda_1\lambda_2\lambda_3^2\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{4\lambda_1^2\lambda_3\lambda_4\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{4\lambda_2^2\lambda_3\lambda_4\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{4\lambda_1\lambda_2\lambda_4^2\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{4\lambda_1\lambda_2^2\lambda_3\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2}-$$

$$\frac{4\lambda_1^2\lambda_2\lambda_4\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_1^2\lambda_2\lambda_3\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_2^3\lambda_3\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_1^3\lambda_4\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_1\lambda_2^2\lambda_4\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_1^3\lambda_3\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2}-$$

$$\frac{2\lambda_1\lambda_2^2\lambda_3\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_1^2\lambda_2\lambda_4\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_2^3\lambda_4\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{2\lambda_1^3\lambda_2\mu_{3,3}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{2\lambda_1\lambda_2^3\mu_{3,3}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{4\lambda_1^2\lambda_2^2\mu_{3,4}}{(\lambda_1^2-\lambda_2^2)^2}-$$

$$\frac{4\lambda_1^2\lambda_2\lambda_3\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{4\lambda_1\lambda_2^2\lambda_4\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{4\lambda_1^2\lambda_2^2\mu_{4,3}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{2\lambda_1^3\lambda_2\mu_{4,4}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{2\lambda_1\lambda_2^3\mu_{4,4}}{(\lambda_1^2-\lambda_2^2)^2},\ \mathtt{l}_1\mathtt{l}_2\to$$

$$\frac{4\lambda_1^2\lambda_3^2\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{8\lambda_1\lambda_2\lambda_3\lambda_4\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{4\lambda_2^2\lambda_4^2\mu_{2,2}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{4\lambda_1^2\lambda_2\lambda_3\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{4\lambda_1\lambda_2^2\lambda_4\mu_{2,3}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_1^3\lambda_3\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_1\lambda_2^2\lambda_3\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2}-$$

$$\frac{2\lambda_1^2\lambda_2\lambda_4\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_2^3\lambda_4\mu_{2,4}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{4\lambda_1^2\lambda_2\lambda_3\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{4\lambda_1\lambda_2^2\lambda_4\mu_{3,2}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{4\lambda_1^2\lambda_2^2\mu_{3,3}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{2\lambda_1^3\lambda_2\mu_{3,4}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{2\lambda_1\lambda_2^3\mu_{3,4}}{(\lambda_1^2-\lambda_2^2)^2}-$$

$$\frac{2\lambda_1^3\lambda_3\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_1\lambda_2^2\lambda_3\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_1^2\lambda_2\lambda_4\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2}-\frac{2\lambda_2^3\lambda_4\mu_{4,2}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{2\lambda_1^3\lambda_2\mu_{4,3}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{2\lambda_1\lambda_2^3\mu_{4,3}}{(\lambda_1^2-\lambda_2^2)^2}+\frac{4\lambda_1^2\lambda_2^2\mu_{4,4}}{(\lambda_1^2-\lambda_2^2)^2},$$

$$\mathtt{l}_1\to-\frac{2\lambda_1\lambda_3\mu_{1,2}}{\lambda_1^2-\lambda_2^2}-\frac{2\lambda_2\lambda_4\mu_{1,2}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_1\lambda_2\mu_{1,3}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_2^2\mu_{1,4}}{\lambda_1^2-\lambda_2^2},\ \mathtt{s}_2\to 0,$$

$$\mathtt{w}_2\to-\frac{2\lambda_2\lambda_3\mu_{2,1}}{\lambda_1^2-\lambda_2^2}-\frac{2\lambda_1\lambda_4\mu_{2,1}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_2^2\mu_{3,1}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_1\lambda_2\mu_{4,1}}{\lambda_1^2-\lambda_2^2},$$

$$\mathtt{l}_2\to-\frac{2\lambda_1\lambda_3\mu_{2,1}}{\lambda_1^2-\lambda_2^2}-\frac{2\lambda_2\lambda_4\mu_{2,1}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_1\lambda_2\mu_{3,1}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_2^2\mu_{4,1}}{\lambda_1^2-\lambda_2^2},\ \to 0\Big\}$$

```
(*We compute a Groebner Basis A and the transformation matrix B:*)
```

In[◦]:=

```
{A, B} = ResourceFunction["ExtendedGroebnerBasis"][eqs,
  {λ3, λ4 , μ1,1, μ1,2, μ1,3, μ1,4, μ2,1, μ2,2, μ2,3, μ2,4 , μ3,1, μ3,2, μ3,3, μ3,4, μ4,1, μ4,2, μ4,3, μ4,4},
  CoefficientDomain → RationalFunctions, MonomialOrder → DegreeReverseLexicographic]
```

Out[◦]=

$$\Big\{\Big\{\Big(\frac{2\lambda_1^2\lambda_2}{\lambda_1^2-\lambda_2^2}-\frac{2\lambda_2^3}{\lambda_1^2-\lambda_2^2}\Big)\mu_{2,4}+\Big(-\frac{2\lambda_1^2\lambda_2}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_2^3}{\lambda_1^2-\lambda_2^2}\Big)\mu_{4,2},\ \cdots 39\cdots,$$

$$\Big(-\frac{4\lambda_1^6\lambda_2^2}{(\lambda_1^2-\lambda_2^2)^2}+\frac{8\lambda_1^4\lambda_2^4}{(\lambda_1^2-\lambda_2^2)^2}-\frac{4\lambda_1^2\lambda_2^6}{(\lambda_1^2-\lambda_2^2)^2}\Big)\lambda_3\lambda_4\mu_{3,3}+\left(\frac{2\lambda_1^6\lambda_2^2}{(\lambda_1^2-\lambda_2^2)^2}-\frac{4\lambda_1^4\lambda_2^4}{(\cdots 1\cdots)^2}+\frac{2\lambda_1^2\lambda_2^6}{(\lambda_1^2-\cdots 1\cdots)^2}\right)\lambda_3^2\mu_{3,4}+$$

$$(\cdots 1\cdots)\lambda_4^2\mu_{3,4}+\cdots 1\cdots+\cdots 1\cdots+\Big(-\frac{4\lambda_1^6\lambda_2^2}{(\lambda_1^2-\lambda_2^2)^2}+\frac{8\lambda_1^4\lambda_2^4}{(\lambda_1^2-\lambda_2^2)^2}-\frac{4\lambda_1^2\lambda_2^6}{(\lambda_1^2-\lambda_2^2)^2}\Big)\lambda_3\lambda_4\mu_{4,4}\Big\},\ \{\cdots 1\cdots\}\Big\}$$

Full expression not available (original memory size: 1.4 MB)

In[∘]:=
```
(*Checking that the calculation is correct:*)
(A // Simplify // Expand) === (B.eqs // Simplify // Expand)
```

Out[∘]=

True

In[∘]:=
```
(*We observe that in fact the elements of the Groebner basis do not have denominators:*)
A // Simplify
```

Out[∘]=

$$\{2\lambda_2(\mu_{2,4}-\mu_{4,2}),\ -2\lambda_2(\mu_{2,3}-\mu_{3,2}),\ -2\lambda_1\lambda_2(-2\lambda_3\mu_{4,2}+\lambda_2(\mu_{3,4}+\mu_{4,3})),$$
$$-2\lambda_2^2(\mu_{3,2}\mu_{4,1}-\mu_{3,1}\mu_{4,2}),\ -2\lambda_2(\mu_{2,2}\mu_{4,1}-\mu_{2,1}\mu_{4,2}),\ -2\lambda_1\lambda_2^2(-2\mu_{3,2}\mu_{4,2}+\mu_{2,2}(\mu_{3,4}+\mu_{4,3})),$$
$$2\lambda_1\lambda_2^2(-2\mu_{3,1}\mu_{4,2}+\mu_{2,1}(\mu_{3,4}+\mu_{4,3})),\ 2\lambda_1\lambda_2^2(-2\mu_{1,3}\mu_{4,2}+\mu_{1,2}(\mu_{3,4}+\mu_{4,3})),$$
$$-4\lambda_1^2\lambda_2^2(\mu_{3,2}^2+\mu_{4,2}^2-\mu_{2,2}(\mu_{3,3}+\mu_{4,4})),\ -4\lambda_1^2\lambda_2^2(\mu_{3,1}\mu_{3,2}+\mu_{4,1}\mu_{4,2}-\mu_{2,1}(\mu_{3,3}+\mu_{4,4})),$$
$$-2\lambda_2^2(\mu_{1,4}\mu_{3,2}-\mu_{1,3}\mu_{4,2}),\ -4\lambda_1^2\lambda_2^2(\mu_{1,3}\mu_{3,2}+\mu_{1,4}\mu_{4,2}-\mu_{1,2}(\mu_{3,3}+\mu_{4,4})),\ -2\lambda_1\lambda_2(-2\lambda_4\mu_{3,2}+\lambda_2(\mu_{3,4}+\mu_{4,3})),$$
$$4\lambda_1^2\lambda_2(\lambda_3\mu_{3,2}+\lambda_4\mu_{4,2}-\lambda_2(\mu_{3,3}+\mu_{4,4})),\ -2\lambda_2(\mu_{2,2}\mu_{3,1}-\mu_{2,1}\mu_{3,2}),\ -2\lambda_2^2(\mu_{1,4}\mu_{3,1}-\mu_{1,3}\mu_{4,1}),$$
$$-2\lambda_2(\lambda_4\mu_{3,1}-\lambda_3\mu_{4,1}),\ -2\lambda_2(\mu_{1,4}\mu_{2,2}-\mu_{1,2}\mu_{4,2}),\ -2\lambda_2(\mu_{1,3}\mu_{2,2}-\mu_{1,2}\mu_{3,2}),$$
$$2\lambda_4\mu_{2,2}-2\lambda_2\mu_{4,2},\ 2\lambda_3\mu_{2,2}-2\lambda_2\mu_{3,2},\ -2\lambda_2(\mu_{1,4}\mu_{2,1}-\mu_{1,2}\mu_{4,1}),\ -2\lambda_2(\mu_{1,3}\mu_{2,1}-\mu_{1,2}\mu_{3,1}),$$
$$2\lambda_4\mu_{2,1}-2\lambda_2\mu_{4,1},\ -2\lambda_3\mu_{2,1}+2\lambda_2\mu_{3,1},\ -2\lambda_2(\lambda_4\mu_{1,3}-\lambda_3\mu_{1,4}),\ 2\lambda_4\mu_{1,2}-2\lambda_2\mu_{1,4},$$
$$-2\lambda_3\mu_{1,2}+2\lambda_2\mu_{1,3},\ -2\lambda_1^2\lambda_2(2\lambda_4\mu_{4,2}^2+\lambda_2(\mu_{3,2}(\mu_{3,4}+\mu_{4,3})-2\mu_{4,2}(\mu_{3,3}+\mu_{4,4}))),$$
$$2\lambda_1^2\lambda_2(2\lambda_4\mu_{4,1}\mu_{4,2}+\lambda_2(\mu_{3,1}(\mu_{3,4}+\mu_{4,3})-2\mu_{4,1}(\mu_{3,3}+\mu_{4,4}))),$$
$$2\lambda_1^2\lambda_2(2\lambda_4\mu_{1,4}\mu_{4,2}+\lambda_2(\mu_{1,3}(\mu_{3,4}+\mu_{4,3})-2\mu_{1,4}(\mu_{3,3}+\mu_{4,4}))),$$
$$-2\lambda_1^2\lambda_2(2\lambda_4^2\mu_{4,2}+\lambda_2(\lambda_3(\mu_{3,4}+\mu_{4,3})-2\lambda_4(\mu_{3,3}+\mu_{4,4}))),\ 4\lambda_1^2\lambda_2^2(-((\mu_{3,1}^2+\mu_{4,1}^2)\mu_{4,2})+\mu_{2,1}\mu_{4,1}(\mu_{3,3}+\mu_{4,4})),$$
$$4\lambda_1^2\lambda_2^2(-((\mu_{1,3}\mu_{3,1}+\mu_{1,4}\mu_{4,1})\mu_{4,2})+\mu_{1,2}\mu_{4,1}(\mu_{3,3}+\mu_{4,4})),$$
$$-2\lambda_1^2\lambda_2^3(\mu_{3,1}^2(\mu_{3,4}+\mu_{4,3})+\mu_{4,1}^2(\mu_{3,4}+\mu_{4,3})-2\mu_{3,1}\mu_{4,1}(\mu_{3,3}+\mu_{4,4})),$$
$$2\lambda_1^2\lambda_2^4(\mu_{1,4}\mu_{4,1}(\mu_{3,4}+\mu_{4,3})+\mu_{1,3}(\mu_{3,1}(\mu_{3,4}+\mu_{4,3})-2\mu_{4,1}(\mu_{3,3}+\mu_{4,4}))),$$
$$2\lambda_1^2\lambda_2^3(\lambda_4\mu_{4,1}(\mu_{3,4}+\mu_{4,3})+\lambda_3(\mu_{3,1}(\mu_{3,4}+\mu_{4,3})-2\mu_{4,1}(\mu_{3,3}+\mu_{4,4}))),$$
$$2\lambda_1^2\lambda_2^3(\mu_{1,3}^2(\mu_{3,4}+\mu_{4,3})+\mu_{1,4}^2(\mu_{3,4}+\mu_{4,3})-2\mu_{1,3}\mu_{1,4}(\mu_{3,3}+\mu_{4,4})),$$
$$4\lambda_1^2\lambda_2^2(-((\mu_{1,3}^2+\mu_{1,4}^2)\mu_{4,2})+\mu_{1,2}\mu_{1,4}(\mu_{3,3}+\mu_{4,4})),$$
$$2\lambda_1^2\lambda_2^3(-\lambda_4\mu_{1,4}(\mu_{3,4}+\mu_{4,3})+\lambda_3(-\mu_{1,3}(\mu_{3,4}+\mu_{4,3})+2\mu_{1,4}(\mu_{3,3}+\mu_{4,4}))),$$
$$2\lambda_1^2\lambda_2^2(\lambda_3^2(\mu_{3,4}+\mu_{4,3})+\lambda_4^2(\mu_{3,4}+\mu_{4,3})-2\lambda_3\lambda_4(\mu_{3,3}+\mu_{4,4}))\}$$

In[∘]:=

(*In order to decompose the coefficients of $[R,\ \kappa\otimes 1\ ]$ not in J⊗J as combination of the elements of A, we use PolynomialReduce. We start with 2 λ_4 $\mu_{1,2}$-2 λ_2 $\mu_{1,4}$, ie the coefficient of w⊗1 : *)

F = PolynomialReduce$[2\lambda_4\mu_{1,2}-2\lambda_2\mu_{1,4}$, A, $\{\lambda_3, \lambda_4, \mu_{1,1}, \mu_{1,2}, \mu_{1,3}, \mu_{1,4}, \mu_{2,1}, \mu_{2,2}, \mu_{2,3}, \mu_{2,4}, \mu_{3,1}, \mu_{3,2}, \mu_{3,3}, \mu_{3,4}, \mu_{4,1}, \mu_{4,2}, \mu_{4,3}, \mu_{4,4}\}]$ // Simplify

Out[∘]=

```
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, 0}
```

(*Indeed this is exactly one of the polynomials in the Groebner basis: *)

In[∘]:=

```
Pick[A, Unitize[F[[1]]], 1] // Simplify // Factor
```

Out[∘]=

$$\{2(\lambda_4\mu_{1,2}-\lambda_2\mu_{1,4})\}$$

(*Let us see, for this case in particular, how the polynomial 2 $(\lambda_4\ \mu_{1,2}-\lambda_2\ \mu_{1,4})$ is obtained from XC0 :*)

```
pos = Flatten[Position[F[[1]], 1]];
B[[pos]].eqs
B[[pos]].eqs // Simplify
```

Out[ ]=

$$\left\{-\lambda_1\left(-\frac{2\lambda_2\lambda_3\mu_{1,2}}{\lambda_1^2-\lambda_2^2}-\frac{2\lambda_1\lambda_4\mu_{1,2}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_2^2\mu_{1,3}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_1\lambda_2\mu_{1,4}}{\lambda_1^2-\lambda_2^2}\right)+\lambda_2\left(-\frac{2\lambda_1\lambda_3\mu_{1,2}}{\lambda_1^2-\lambda_2^2}-\frac{2\lambda_2\lambda_4\mu_{1,2}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_1\lambda_2\mu_{1,3}}{\lambda_1^2-\lambda_2^2}+\frac{2\lambda_2^2\mu_{1,4}}{\lambda_1^2-\lambda_2^2}\right)\right\}$$

Out[ ]=

$$\{2\lambda_4\mu_{1,2}-2\lambda_2\mu_{1,4}\}$$

(*We now check the remaining coefficients of $[R, \kappa\otimes 1]$ not in J⊗J :*)

In[ ]:=

$$\text{PolynomialReduce}[2\lambda_3\mu_{1,2}-2\lambda_2\mu_{1,3}, A, \{\lambda_3, \lambda_4, \mu_{1,1}, \mu_{1,2}, \mu_{1,3}, \mu_{1,4}, \mu_{2,1}, \mu_{2,2}, \mu_{2,3}, \mu_{2,4}, \mu_{3,1}, \mu_{3,2}, \mu_{3,3}, \mu_{3,4}, \mu_{4,1}, \mu_{4,2}, \mu_{4,3}, \mu_{4,4}\}] \text{ // Simplify}$$

$$\text{PolynomialReduce}[2\lambda_4\mu_{2,2}-2\lambda_2\mu_{2,4}, A, \{\lambda_3, \lambda_4, \mu_{1,1}, \mu_{1,2}, \mu_{1,3}, \mu_{1,4}, \mu_{2,1}, \mu_{2,2}, \mu_{2,3}, \mu_{2,4}, \mu_{3,1}, \mu_{3,2}, \mu_{3,3}, \mu_{3,4}, \mu_{4,1}, \mu_{4,2}, \mu_{4,3}, \mu_{4,4}\}] \text{ // Simplify}$$

$$\text{PolynomialReduce}[2\lambda_3\mu_{2,2}-2\lambda_2\mu_{2,3}, A, \{\lambda_3, \lambda_4, \mu_{1,1}, \mu_{1,2}, \mu_{1,3}, \mu_{1,4}, \mu_{2,1}, \mu_{2,2}, \mu_{2,3}, \mu_{2,4}, \mu_{3,1}, \mu_{3,2}, \mu_{3,3}, \mu_{3,4}, \mu_{4,1}, \mu_{4,2}, \mu_{4,3}, \mu_{4,4}\}] \text{ // Simplify}$$

Out[ ]=

```
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, 0}
```

Out[ ]=

```
{{-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, 0}
```

Out[ ]=

```
{{0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, 0}
```

(*Now we turn into $[R, 1\otimes\kappa]$:*)

In[ ]:=

```
vari = {s1, w1, l1, s2, w2, l2};
variL = {"s1", "w1", "l1", "s2", "w2", "l2"};
p = (R1,2 κ3 // m2,3→2) - (R1,2 κ3 // m3,2→2);
cr = CoefficientRules[p, vari];
convert[{v__Integer}] := StringJoin@Pick[variL, {v}, 1];
(convert[First[#]] → Last[#]) & /@ cr
eqs = Thread[Values[cr] == 0];
```

Out[ ]=

$$\{s_1w_2 \to 2\lambda_4\mu_{2,2}-2\lambda_2\mu_{4,2},\ s_1l_2 \to 2\lambda_3\mu_{2,2}-2\lambda_2\mu_{3,2},\ w_1w_2 \to 2\lambda_4\mu_{2,3}-2\lambda_2\mu_{4,3},\ w_1l_2 \to 2\lambda_3\mu_{2,3}-2\lambda_2\mu_{3,3},\ l_1w_2 \to 2\lambda_4\mu_{2,4}-2\lambda_2\mu_{4,4},\ l_1l_2 \to 2\lambda_3\mu_{2,4}-2\lambda_2\mu_{3,4},\ w_2 \to 2\lambda_4\mu_{2,1}-2\lambda_2\mu_{4,1},\ l_2 \to 2\lambda_3\mu_{2,1}-2\lambda_2\mu_{3,1}\}$$

(*We check the coefficients that do not belong to J ⊗ J:*)

```
In[∘]:= PolynomialReduce[2 λ4 μ2,2 - 2 λ2 μ4,2, A, {λ3, λ4 , μ1,1, μ1,2, μ1,3, μ1,4,
          μ2,1, μ2,2, μ2,3, μ2,4 , μ3,1, μ3,2, μ3,3, μ3,4, μ4,1, μ4,2, μ4,3, μ4,4}] // Simplify
        PolynomialReduce[2 λ3 μ2,2 - 2 λ2 μ3,2, A, {λ3, λ4 , μ1,1, μ1,2, μ1,3, μ1,4,
          μ2,1, μ2,2, μ2,3, μ2,4 , μ3,1, μ3,2, μ3,3, μ3,4, μ4,1, μ4,2, μ4,3, μ4,4}] // Simplify
        PolynomialReduce[2 λ4 μ2,1 - 2 λ2 μ4,1, A, {λ3, λ4 , μ1,1, μ1,2, μ1,3, μ1,4,
          μ2,1, μ2,2, μ2,3, μ2,4 , μ3,1, μ3,2, μ3,3, μ3,4, μ4,1, μ4,2, μ4,3, μ4,4}] // Simplify
        PolynomialReduce[2 λ3 μ2,1 - 2 λ2 μ3,1, A, {λ3, λ4 , μ1,1, μ1,2, μ1,3, μ1,4,
          μ2,1, μ2,2, μ2,3, μ2,4 , μ3,1, μ3,2, μ3,3, μ3,4, μ4,1, μ4,2, μ4,3, μ4,4}] // Simplify

Out[∘]=
        {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, 0}

Out[∘]=
        {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, 0}

Out[∘]=
        {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, 0}

Out[∘]=
        {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, 0}

        (*This concludes the proof.*)
```