Examples:

In[ ]:=
```
(*We establish the multiplication for SW. We
 set the variable l to be sw. The bar denotes inverse. *)
vars = {s, w, l};
m_{i_,j_->k_}[Z_] := Expand[Z] /. {s_i s_j -> 1, w_i w_j -> 0, s_i w_j -> l_k, w_i s_j -> -l_k, l_i s_j -> -w_k, s_i l_j -> w_k,
    l_i w_j -> 0, w_i l_j -> 0, l_i l_j -> 0, x__i /; MemberQ[vars, x] :> x_k, x__j /; MemberQ[vars, x] :> x_k}
```

In[ ]:=
```
(*We establish the multiplication for SW. We
 set the variable l to be sw. The bar denotes inverse. *)
vars = {s, w, l};
m_{i_,j_->k_}[Z_] := Expand[Z] /. {s_i s_j -> 1, w_i w_j -> 0, s_i w_j -> l_k, w_i s_j -> -l_k, l_i s_j -> -w_k, s_i l_j -> w_k,
    l_i w_j -> 0, w_i l_j -> 0, l_i l_j -> 0, x__i /; MemberQ[vars, x] :> x_k, x__j /; MemberQ[vars, x] :> x_k}
```

In[ ]:=
```
(* We start by declaring the elements R and κ: *)
κ_{i_} := λ1 + λ2 s_i + λ3 w_i + λ4 l_i;
κbar_{i_} := λbar1 + λbar2 s_i + λbar3 w_i + λbar4 l_i;
R_{i_,j_} := μ1,1 + μ1,2 s_i + μ1,3 w_i + μ1,4 l_i + μ2,1 s_j + μ2,2 s_i s_j + μ2,3 w_i s_j + μ2,4 l_i s_j +
   μ3,1 w_j + μ3,2 s_i w_j + μ3,3 w_i w_j + μ3,4 l_i w_j + μ4,1 l_j + μ4,2 s_i l_j + μ4,3 w_i l_j + μ4,4 l_i l_j
Rbar_{i_,j_} := μbar1,1 + μbar1,2 s_i + μbar1,3 w_i + μbar1,4 l_i + μbar2,1 s_j + μbar2,2 s_i s_j + μbar2,3 w_i s_j + μbar2,4 l_i s_j +
   μbar3,1 w_j + μbar3,2 s_i w_j + μbar3,3 w_i w_j + μbar3,4 l_i w_j + μbar4,1 l_j + μbar4,2 s_i l_j + μbar4,3 w_i l_j + μbar4,4 l_i l_j
(*We also implement the element v :*)
v_{i_} := R1,3 κbar2 // m1,2->1 // m1,3->i
vbar_{i_} := Rbar3,1 κbar2 // m1,2->1 // m1,3->i
```

In[ ]:=
```
(*By the previous lemma, we simply set the values of the bar variables: *)
{λbar1, λbar2, λbar3, λbar4} = {λ1/(λ1^2 - λ2^2), -λ2/(λ1^2 - λ2^2), -λ3/(λ1^2 - λ2^2), -λ4/(λ1^2 - λ2^2)};
rulesRinv =
  {μbar1,1 -> -((-μ1,1^3 + μ1,1 μ1,2^2 + μ1,1 μ2,1^2 - 2 μ1,2 μ2,1 μ2,2 + μ1,1 μ2,2^2) / (μ1,1^4 - 2 μ1,1^2 μ1,2^2 + μ1,2^4 - 2 μ1,1^2 μ2,1^2 -
          2 μ1,2^2 μ2,1^2 + μ2,1^4 + 8 μ1,1 μ1,2 μ2,1 μ2,2 - 2 μ1,1^2 μ2,2^2 - 2 μ1,2^2 μ2,2^2 - 2 μ2,1^2 μ2,2^2 + μ2,2^4)),
   μbar1,2 -> -((μ1,1^2 μ1,2 - μ1,2^3 + μ1,2 μ2,1^2 - 2 μ1,1 μ2,1 μ2,2 + μ1,2 μ2,2^2) / (μ1,1^4 - 2 μ1,1^2 μ1,2^2 + μ1,2^4 -
          2 μ1,1^2 μ2,1^2 - 2 μ1,2^2 μ2,1^2 + μ2,1^4 + 8 μ1,1 μ1,2 μ2,1 μ2,2 - 2 μ1,1^2 μ2,2^2 - 2 μ1,2^2 μ2,2^2 - 2 μ2,1^2 μ2,2^2 + μ2,2^4)),
   μbar1,3 -> -((μ1,1^2 μ1,3 - μ1,2^2 μ1,3 + μ1,3 μ2,1^2 - μ1,3 μ2,2^2 - 2 μ1,1 μ2,1 μ2,3 + 2 μ1,2 μ2,2 μ2,3) /
       (μ1,1^4 - 2 μ1,1^2 μ1,2^2 + μ1,2^4 - 2 μ1,1^2 μ2,1^2 - 2 μ1,2^2 μ2,1^2 + μ2,1^4 +
          8 μ1,1 μ1,2 μ2,1 μ2,2 - 2 μ1,1^2 μ2,2^2 - 2 μ1,2^2 μ2,2^2 - 2 μ2,1^2 μ2,2^2 + μ2,2^4)), μbar1,4 ->
    -((μ1,1^2 μ1,4 - μ1,2^2 μ1,4 + μ1,4 μ2,1^2 - μ1,4 μ2,2^2 - 2 μ1,1 μ2,1 μ2,4 + 2 μ1,2 μ2,2 μ2,4) / (μ1,1^4 - 2 μ1,1^2 μ1,2^2 + μ1,2^4 -
          2 μ1,1^2 μ2,1^2 - 2 μ1,2^2 μ2,1^2 + μ2,1^4 + 8 μ1,1 μ1,2 μ2,1 μ2,2 - 2 μ1,1^2 μ2,2^2 - 2 μ1,2^2 μ2,2^2 - 2 μ2,1^2 μ2,2^2 + μ2,2^4)),
   μbar2,1 -> -((μ1,1^2 μ2,1 + μ1,2^2 μ2,1 - μ2,1^3 - 2 μ1,1 μ1,2 μ2,2 + μ2,1 μ2,2^2) / (μ1,1^4 - 2 μ1,1^2 μ1,2^2 + μ1,2^4 -
          2 μ1,1^2 μ2,1^2 - 2 μ1,2^2 μ2,1^2 + μ2,1^4 + 8 μ1,1 μ1,2 μ2,1 μ2,2 - 2 μ1,1^2 μ2,2^2 - 2 μ1,2^2 μ2,2^2 - 2 μ2,1^2 μ2,2^2 + μ2,2^4)),
   μbar2,2 -> -((-2 μ1,1 μ1,2 μ2,1 + μ1,1^2 μ2,2 + μ1,2^2 μ2,2 + μ2,1^2 μ2,2 - μ2,2^3) / (μ1,1^4 - 2 μ1,1^2 μ1,2^2 + μ1,2^4 -
          2 μ1,1^2 μ2,1^2 - 2 μ1,2^2 μ2,1^2 + μ2,1^4 + 8 μ1,1 μ1,2 μ2,1 μ2,2 - 2 μ1,1^2 μ2,2^2 - 2 μ1,2^2 μ2,2^2 - 2 μ2,1^2 μ2,2^2 + μ2,2^4)),
   μbar2,3 -> -((-2 μ1,1 μ1,3 μ2,1 + 2 μ1,2 μ1,3 μ2,2 + μ1,1^2 μ2,3 - μ1,2^2 μ2,3 + μ2,1^2 μ2,3 - μ2,2^2 μ2,3) /
```

$$\left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right),\ \overline{\mu}_{2,4} \to -\left(\left(-2\,\mu_{1,1}\,\mu_{1,4}\,\mu_{2,1} + 2\,\mu_{1,2}\,\mu_{1,4}\,\mu_{2,2} + \mu_{1,1}^2\,\mu_{2,4} - \mu_{1,2}^2\,\mu_{2,4} + \mu_{2,1}^2\,\mu_{2,4} - \mu_{2,2}^2\,\mu_{2,4}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\right),$$

$$\overline{\mu}_{3,1} \to -\left(\left(\mu_{1,1}^2\,\mu_{3,1} + \mu_{1,2}^2\,\mu_{3,1} - \mu_{2,1}^2\,\mu_{3,1} - \mu_{2,2}^2\,\mu_{3,1} - 2\,\mu_{1,1}\,\mu_{1,2}\,\mu_{3,2} + 2\,\mu_{2,1}\,\mu_{2,2}\,\mu_{3,2}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\right),\ \overline{\mu}_{3,2} \to -\left(\left(-2\,\mu_{1,1}\,\mu_{1,2}\,\mu_{3,1} + 2\,\mu_{2,1}\,\mu_{2,2}\,\mu_{3,1} + \mu_{1,1}^2\,\mu_{3,2} + \mu_{1,2}^2\,\mu_{3,2} - \mu_{2,1}^2\,\mu_{3,2} - \mu_{2,2}^2\,\mu_{3,2}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\right),$$

$$\overline{\mu}_{3,3} \to -\left(\left(-2\,\mu_{1,1}\,\mu_{1,3}\,\mu_{3,1} + 2\,\mu_{2,1}\,\mu_{2,3}\,\mu_{3,1} + 2\,\mu_{1,2}\,\mu_{1,3}\,\mu_{3,2} - 2\,\mu_{2,2}\,\mu_{2,3}\,\mu_{3,2} + \mu_{1,1}^2\,\mu_{3,3} - \mu_{1,2}^2\,\mu_{3,3} - \mu_{2,1}^2\,\mu_{3,3} + \mu_{2,2}^2\,\mu_{3,3} + 2\,\mu_{1,4}\,\mu_{2,2}\,\mu_{4,1} - 2\,\mu_{1,2}\,\mu_{2,4}\,\mu_{4,1} - 2\,\mu_{1,4}\,\mu_{2,1}\,\mu_{4,2} + 2\,\mu_{1,1}\,\mu_{2,4}\,\mu_{4,2} + 2\,\mu_{1,2}\,\mu_{2,1}\,\mu_{4,4} - 2\,\mu_{1,1}\,\mu_{2,2}\,\mu_{4,4}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\right),$$

$$\overline{\mu}_{3,4} \to -\left(\left(-2\,\mu_{1,1}\,\mu_{1,4}\,\mu_{3,1} + 2\,\mu_{2,1}\,\mu_{2,4}\,\mu_{3,1} + 2\,\mu_{1,2}\,\mu_{1,4}\,\mu_{3,2} - 2\,\mu_{2,2}\,\mu_{2,4}\,\mu_{3,2} + \mu_{1,1}^2\,\mu_{3,4} - \mu_{1,2}^2\,\mu_{3,4} - \mu_{2,1}^2\,\mu_{3,4} + \mu_{2,2}^2\,\mu_{3,4} + 2\,\mu_{1,3}\,\mu_{2,2}\,\mu_{4,1} - 2\,\mu_{1,2}\,\mu_{2,3}\,\mu_{4,1} - 2\,\mu_{1,3}\,\mu_{2,1}\,\mu_{4,2} + 2\,\mu_{1,1}\,\mu_{2,3}\,\mu_{4,2} + 2\,\mu_{1,2}\,\mu_{2,1}\,\mu_{4,3} - 2\,\mu_{1,1}\,\mu_{2,2}\,\mu_{4,3}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\right),$$

$$\overline{\mu}_{4,1} \to -\left(\left(\mu_{1,1}^2\,\mu_{4,1} + \mu_{1,2}^2\,\mu_{4,1} - \mu_{2,1}^2\,\mu_{4,1} - \mu_{2,2}^2\,\mu_{4,1} - 2\,\mu_{1,1}\,\mu_{1,2}\,\mu_{4,2} + 2\,\mu_{2,1}\,\mu_{2,2}\,\mu_{4,2}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\right),\ \overline{\mu}_{4,2} \to -\left(\left(-2\,\mu_{1,1}\,\mu_{1,2}\,\mu_{4,1} + 2\,\mu_{2,1}\,\mu_{2,2}\,\mu_{4,1} + \mu_{1,1}^2\,\mu_{4,2} + \mu_{1,2}^2\,\mu_{4,2} - \mu_{2,1}^2\,\mu_{4,2} - \mu_{2,2}^2\,\mu_{4,2}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\right),$$

$$\overline{\mu}_{4,3} \to -\left(\left(2\,\mu_{1,4}\,\mu_{2,2}\,\mu_{3,1} - 2\,\mu_{1,2}\,\mu_{2,4}\,\mu_{3,1} - 2\,\mu_{1,4}\,\mu_{2,1}\,\mu_{3,2} + 2\,\mu_{1,1}\,\mu_{2,4}\,\mu_{3,2} + 2\,\mu_{1,2}\,\mu_{2,1}\,\mu_{3,4} - 2\,\mu_{1,1}\,\mu_{2,2}\,\mu_{3,4} - 2\,\mu_{1,1}\,\mu_{1,3}\,\mu_{4,1} + 2\,\mu_{2,1}\,\mu_{2,3}\,\mu_{4,1} + 2\,\mu_{1,2}\,\mu_{1,3}\,\mu_{4,2} - 2\,\mu_{2,2}\,\mu_{2,3}\,\mu_{4,2} + \mu_{1,1}^2\,\mu_{4,3} - \mu_{1,2}^2\,\mu_{4,3} - \mu_{2,1}^2\,\mu_{4,3} + \mu_{2,2}^2\,\mu_{4,3}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\right),$$

$$\overline{\mu}_{4,4} \to -\left(\left(2\,\mu_{1,3}\,\mu_{2,2}\,\mu_{3,1} - 2\,\mu_{1,2}\,\mu_{2,3}\,\mu_{3,1} - 2\,\mu_{1,3}\,\mu_{2,1}\,\mu_{3,2} + 2\,\mu_{1,1}\,\mu_{2,3}\,\mu_{3,2} + 2\,\mu_{1,2}\,\mu_{2,1}\,\mu_{3,3} - 2\,\mu_{1,1}\,\mu_{2,2}\,\mu_{3,3} - 2\,\mu_{1,1}\,\mu_{1,4}\,\mu_{4,1} + 2\,\mu_{2,1}\,\mu_{2,4}\,\mu_{4,1} + 2\,\mu_{1,2}\,\mu_{1,4}\,\mu_{4,2} - 2\,\mu_{2,2}\,\mu_{2,4}\,\mu_{4,2} + \mu_{1,1}^2\,\mu_{4,4} - \mu_{1,2}^2\,\mu_{4,4} - \mu_{2,1}^2\,\mu_{4,4} + \mu_{2,2}^2\,\mu_{4,4}\right) \Big/ \left(\mu_{1,1}^4 - 2\,\mu_{1,1}^2\,\mu_{1,2}^2 + \mu_{1,2}^4 - 2\,\mu_{1,1}^2\,\mu_{2,1}^2 - 2\,\mu_{1,2}^2\,\mu_{2,1}^2 + \mu_{2,1}^4 + 8\,\mu_{1,1}\,\mu_{1,2}\,\mu_{2,1}\,\mu_{2,2} - 2\,\mu_{1,1}^2\,\mu_{2,2}^2 - 2\,\mu_{1,2}^2\,\mu_{2,2}^2 - 2\,\mu_{2,1}^2\,\mu_{2,2}^2 + \mu_{2,2}^4\right)\right)\Big\};$$

```
Scan[Set @@ # &, rulesRinv]
```

```
(*To change from one test to another, please exit the kernel.*)
```

TEST 1:

In[•]:=
```
μ1,3 = 0; μ2,4 = 1; μ3,1 = 1; λ1 = 0; μ4,4 = 1; μ4,1 = 1; λ2 = -1; λ3 = -1; λ4 = -1; μ1,2 = 0;
μ1,4 = 0; μ2,1 = 1; μ2,2 = 1; μ2,3 = 1; μ3,2 = 1; μ3,3 = 1; μ3,4 = 1; μ4,2 = 1; μ4,3 = 1;
```

TEST 2:

In[ ]:=
$$\lambda_1 = 1;\ \lambda_2 = 0;\ \lambda_3 = \frac{2\,\mu_{2,3}}{\mu_{1,1}};\ \lambda_4 = -\frac{2\,\mu_{2,3}}{\mu_{1,1}};\ \mu_{1,2} = 0;\ \mu_{2,1} = 0;\ \mu_{2,2} = 0;$$
$$\mu_{1,3} = -\mu_{1,4};\ \mu_{2,4} = -\mu_{2,3};\ \mu_{3,1} = \mu_{1,4};\ \mu_{3,2} = -\mu_{2,3};\ \mu_{4,1} = -\mu_{1,4};$$
$$\mu_{4,2} = \mu_{2,3};\ \mu_{4,3} = \frac{2\,\mu_{1,4}^2 - 2\,\mu_{2,3}^2 - \mu_{1,1}\,\mu_{3,4}}{\mu_{1,1}};\ \mu_{4,4} = \frac{-2\,\mu_{1,4}^2 + 2\,\mu_{2,3}^2 - \mu_{1,1}\,\mu_{3,3}}{\mu_{1,1}};$$

TEST 3:

In[ ]:=
$$\lambda_1 = 1;\ \lambda_2 = 0;\ \mu_{1,2} = 0;\ \mu_{2,1} = 0;\ \mu_{2,2} = 0;\ \lambda_3 = \frac{2\,\mu_{2,3}}{\mu_{1,1}};\ \lambda_4 = -\frac{2\,\mu_{2,3}}{\mu_{1,1}};$$
$$\mu_{1,3} = -\mu_{1,4};\ \mu_{2,4} = -\mu_{2,3};\ \mu_{3,1} = \mu_{1,4};\ \mu_{3,2} = -\mu_{2,3};\ \mu_{4,1} = -\mu_{1,4};$$
$$\mu_{4,2} = \mu_{2,3};\ \mu_{4,3} = \frac{2\,\mu_{1,4}^2 - 2\,\mu_{2,3}^2 - \mu_{1,1}\,\mu_{3,4}}{\mu_{1,1}};\ \mu_{4,4} = \frac{-2\,\mu_{1,4}^2 + 2\,\mu_{2,3}^2 - \mu_{1,1}\,\mu_{3,3}}{\mu_{1,1}};$$

Checking that each of these examples indeed defines an XC-structure on SW:

In[ ]:=
```
(R̄[1,2] R[3,4] // m[1,3→1] // m[2,4→2]) - 1 // Simplify
((R[1,2] R[4,3] R[5,6] // m[1,4→1] // m[2,5→2] // m[3,6→3]) - (R[1,6] R[2,3] R[4,5] // m[1,4→1] // m[2,5→2] // m[3,6→3]))
((κ[1] κ[2] R[3,4] κ̄[5] κ̄[6] // m[1,3→1] // m[1,5→1] // m[2,4→2] // m[2,6→2]) - R[1,2]) // Simplify
((κ[1] κ[2] R̄[3,4] κ̄[5] κ̄[6] // m[1,3→1] // m[1,5→1] // m[2,4→2] // m[2,6→2]) - R̄[1,2]) // Simplify
((R̄[1,4] R[3,2] κ[6] // m[1,3→1] // m[2,6→2] // m[2,4→2]) - κ[2]) // Expand // Simplify
((R̄[5,1] R[2,3] κ̄[4] // m[1,3→1] // m[2,4→2] // m[2,5→2]) - κ̄[2]) // Expand // Simplify
(κ[1] κ̄[2] // m[1,2→1]) - 1 // Simplify // Expand
((R[1,3] κ̄[2] // m[1,2→1] // m[1,3→1]) - (R[3,1] κ[2] // m[1,2→1] // m[1,3→1])) // Expand
```

Out[ ]=
0

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Out[ ]=
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Out[ ]=
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```
Once[AppendTo[$Path, FileNameJoin[{"/Users/usename/.../MathematicaNotebooks/KnotTheory"}]];
  << KnotTheory` ;
 ];
```

Get: Cannot open KnotTheory`.

In[ ]:=
```
RVK::usage =
  "RVK[xs, rots] represents a Rotational Virtual Knot with a list of n Xp/Xm crossings
    xs and a length 2n list of rotation numbers rots. Crossing
    sites are indexed 1 through 2n, and rots[[k]] is the rotation
    between site k-1 and site k. RVK is also a casting operator
    converting to the RVK presentation from other knot presentations.";
```

In[ ]:=
```
RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :> { Xp[x[[4]], x[[1]]] PositiveQ@x
                           Xm[x[[2]], x[[1]]]      True    ];
   For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
     front = Flatten@Replace[front, k → (xs /. {
              Xp[k + 1, l_] | Xm[l_, k + 1] :> {l, k + 1, 1 - l},
              Xp[l_, k + 1] | Xm[k + 1, l_] :> (++rots[[l]]; {1 - l, k + 1, l}),
              _Xp | _Xm :> {}
             }), {1}],
     Cases[front, k | -k] /. {k, -k} :> --rots[[k + 1]];
    ]];
   RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[ ]:=
```
rot[i_, 0] := 1;
rot[i_, n_] := rot[i, n];
rot[i_, n_] := Module[{j},
   rot[i, n] = If[n > 0, rot[i, n - 1] κ_j, rot[i, n + 1] κ̄_j] // m_{i,j→i}];
```

In[ ]:=
```
Width[pd_PD] := Max[Length /@ FoldList[Complement[#1 ∪ #2, #1 ∩ #2] &, {}, List @@ List @@@ pd]]
```

In[ ]:=
```
ThinPosition[K_] := Module[{todo, done, pd, cc},
   todo = List @@ PD@K; done = {}; pd = PD[];
   While[todo =!= {},
    AppendTo[pd, cc = RandomChoice@MaximalBy[todo, Length[done ∩ List @@ #] &]];
    todo = DeleteCases[todo, cc];
    done = done ∪ List @@ cc];
   pd ];
ThinPosition[K_, n_] := First@MinimalBy[Table[ThinPosition[K], n], Width];
```

In[ ]:=
```
Z[K_] := Z[RVK@ThinPosition[K, 100]];
Z[rvk_RVK] := Monitor[ Module[{ζ, done, st, cc, χ, i, j, k},
   ζ = 1; done = {}; st = Range[2 Length[rvk[[1]]]]; $M = {};
   Do[AppendTo[$M, cc];
    {i, j} = List @@ cc;
    χ = (cc /. {_Xp :> R_{i,j} v̄_0, _Xm :> R̄_{i,j} v_0}) // m_{j,0→j};
    Do[χ = (rot[0, rvk[[2, k]]] χ) // m_{0,k→k}, {k, {i, j}}];
    ζ *= χ;
    Do[
     If[MemberQ[done, k + 1], ζ = ζ // m_{k,k+1→k}; st = st /. k + 1 → k];
     If[MemberQ[done, k - 1], ζ = ζ // m_{st[[k-1]],k→st[[k-1]]}; st = st /. k → st[[k - 1]],
     {k, {i, j}}];
    done = done ∪ {i, j},
    {cc, rvk[[1]]}
   ];
   ζ
  ], {Length@$M, $M}]
```

In[ ]:=
```
{#, Z[#] // Simplify} & /@ AllKnots[{1, 7}] // MatrixForm
```

Get: ParentDirectory[File] in $Path is not a string.

KnotTheory: Loading precomputed data in PD4Knots`.

Out[ ]//MatrixForm=

Knot[3, 1]	1
Knot[4, 1]	1
Knot[5, 1]	1
Knot[5, 2]	1
Knot[6, 1]	1
Knot[6, 2]	1
Knot[6, 3]	1
Knot[7, 1]	1
Knot[7, 2]	1
Knot[7, 3]	1
Knot[7, 4]	1
Knot[7, 5]	1
Knot[7, 6]	1
Knot[7, 7]	1