R. van der Veen
J. Becerra

Presentations - Topics in Topology

Below there is a list of projects that you will have to present on the days 28 March 3-5pm, 30 March 3-5pm, 3 April 1-3pm and 4 April 3-5pm. There will be three presentations each day. Presentations should last 20 minutes. Together with the presentation, you are expected to prepare a LaTeX-ed handout to be brought to your presentation. The handout is expected to be self-contained and detailed (e.g. you are allowed to sketch a proof during your presentation, but the handout should contain a full proof). Handouts completed after the presentation will not be taken into consideration. During your peers' presentations, you are expected to ask questions and comments (this will count for participation).

Please send a list with *at least* two presentations **before 23 March** to j.becerra@rug.nl, indicating the order of preference. Include also a list with your preferred days (*at least* two as well).

OP=OP! The webpage will be updated with the presentations and dates already chosen. We strongly advise you to send your choice as soon as possible.

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1 The HOMFLY-PT polynomial

In this talk you will explore a third link polynomial invariant, called the *HOMFLY-PT polynomial*. This is a 2-variable polynomial

$$P : \mathcal{L} \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$$

that is uniquely determined by the condition $P_O = 1$ and the skein relation

$$xP_{L_+} + x^{-1}P_{L_-} + yP_{L_0} = 0,$$

where (L_+, L_-, L_0) denotes the usual triple.

You are expected to touch upon the following items:

1. State the theorem of existence and uniqueness of the HOMFLY-PT polynomial. You can omit the existence part, but you should prove the uniqueness part. *Hint.* If L denotes the trivial n-component link then $P_L(x, y) = (-(x + x^{-1})y^{-1})^{n-1}$.

2. The HOMFLY-PT polynomial recovers the Jones and Alexander polynomials, more particularly
$$J_L(t) = P_L(\mathbf{i}t^{-1}, \mathbf{i}(t^{-1/2} - t^{1/2})) \quad , \quad \Delta_L(t) = P_L(\mathbf{i}, \mathbf{i}(t^{1/2} - t^{-1/2})),$$
where $\mathbf{i}$ denotes the imaginary unit.

3. Compute at least one example.

4. Is the HOMFLY-PT polynomial a complete knot invariant?

5. What is its behaviour under orientation reversal, mirror images and connected sums?

6. Use the HOMFLY-PT polynomial to show that for any link L,
$$J_L(-1) = \Delta_L(-1).$$

References:

- *An Introduction to Knot Theory*, WBR Lickorish, (chapter 15 and 16),
- *Knots*, Burde Zieschang. (chapter 16)
- *A new polynomial invariant of knots and links.* Freyd, P., Yetter, D., Hoste, J., Lickorish, W.R., Millett, K. and Ocneanu, A.,

2 Braids

In this project you will study a very particular type of XC-tangle where there are no C's. Braids are particularly useful as it turns out that any link arise from a braid, so to study knot and link invariants we can sometimes focus on braids.

1. Define (oriented) braids topologically as an embedding of intervals in a cube with no critical points with respect to the vertical coordinate, modulo isotopy rel. endpoints. Show that stacking gives rise to a group B_n for every n, called the *braid group*. What is the inverse of an element? Why is the product associative?

2. Give a presentation of B_n (Artin's theorem, discuss the proof).

3. Explain how any braid gives rise to a link by taking the *closure cl* of the braid. Write down an algebraic formula of the closure map in the language of XC-tangles.

4. When does this closure produce a knot?

5. Show the Alexander theorem, saying that the closure map cl is surjective, that is, that every link is the closure of some braid. What is the source of this closure map $cl : (?) \to \mathcal{L}$?

6. State the Markov theorem, saying exactly when two closures of two braids are the same link. Show the easy "if" part only.

7. Make a connection between OU-tangle diagrams (allowing C's) and braids.

References:

- *Knot, Links, Braids and 3-manifolds,* Prasolov and Sossinsky (chapter 3)
- *Quantum Invariants, a study of knots, 3-manifolds and their sets,* T. Ohtsuki. (§2.1)
- *Knots and Physics*, LH Kauffman (chapter 7)
- *Over then under tangles*, Bar-Natan, Dancso, van der Veen https://arxiv.org/abs/2007.09828v4

3 The Jones' Jones polynomial definition.

In this project you will go back to the late 1980s and study the original definition that Vaughan Jones gave of the celebrated link polynomial invariant that we know these days as the Jones polynomial.

1. Breafly explain the braid group B_n and state the Alexander and Markov theorems, and consider the standard presentation of B_n (Artin's presentation) with generators $\sigma_1, \ldots, \sigma_{n-1}$.

2. Define what a Markov trace $(J_n : B_n \to k)_{n\geq 2}$ is and prove that any Markov trace gives rise to a link invariant $J : \mathcal{L} \to k$.

3. For $n \geq 0$, consider the set D_n of diagrams of two parallel lines with n marked points on each line and points connected by planar curves lying in between the two lines with no crossings allowed. Let k be a ring and let $\tau \in k$, and define $TL(n, \tau) := k[D_n]$, the *Temperley-Lieb algebra*. Explain the algebra structure.

4. Give a presentation of the Temperley-Lieb algebra in terms of generators e_i and relations.

5. Now consider $k = \mathbb{Z}[A, A^{-1}]$ and $\tau := -A^2 - A^{-2}$. Sketch that the map

$$\rho : B_n \to TL(n, \tau) \qquad , \qquad \rho(\sigma_i^{\pm}) := A^{\pm 1} \cdot 1 + A^{\mp 1} e_i$$

   is a group homomorphism.

6. Consider the map

$$tr : TL(n, \tau) \to k \qquad , \qquad tr(X) := (-A^{-2} - A^2)^{c-n}$$

   where c is the number of components of $cl(X)$. Show that it is a Markov trace.

7. Conclude that for a braid $\beta \in B_n$ we have something like

$$J_{cl(\beta)}(t) = (-A)^{3w(\beta)}(-A^{-2} - A^2)^{n-1} tr(\rho(\beta))_{|t^{1/2}=A^{-2}}$$

   (check the details of this formula!)

8. Elaborate on the significance of Temperley-Lieb algebra either from the point of view of statistical mechanics or by showing how it can be used to also include the colored Jones polynomials (coming from higher dimensional representations of quantum sl_2).

References:

- *Knots and Physics,* LH Kauffman.
- *The Jones Polynomial,* Vaughan FR Jones (2005, available on Jones website).
- *Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds,* LH Kauffman and S Lins

4 The Alexander polynomial via Fox calculus

In this talk you will construct the Alexander polynomial of a knot K in $\mathbb{R}^3$ out of the fundamental group presentation $\mathcal{W}(K)$ using a tool called *Fox calculus.* If $F_n = F(x_1, \ldots, x_n)$ denotes the free group on n variables $x_1, \ldots, x_n$, then the Fox calculus studies formal derivatives

$$\frac{\partial}{\partial x_i} : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n],$$

where $\mathbb{Z}[F_n]$ is the free abelian group on F_n.

This is an outline of what this project should cover:

1. Define
$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \quad , \quad \frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij} x_i^{-1} \quad , \quad \frac{\partial (uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} + u \frac{\partial v}{\partial x_j}$$
for $u, v \in F_n$. Why does this give a well-defined map
$$\frac{\partial}{\partial x_j} : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$$
2. Let G be a group presented as $G \cong \langle x_1, \ldots, x_n | r_1, \ldots, r_s \rangle$. Define the Alexander matrix A of this presentation. A should be an $s \times n$ matrix with entries in $\mathbb{Z}[G^{ab}]$, where $G^{ab} := G/[G, G]$ is its abelianisation.
3. Let $\mathcal{W}(D)$ be the fundamental group presentation of a diagram D of a knot K in $\mathbb{R}^3$. Show that $\mathcal{W}(D)^{ab} \cong \mathbb{Z}$.
4. Show that if $G = \mathcal{W}(D)$ for a knot diagram D then the Alexander matrix A is has entries in $\mathbb{Z}[t, t^{-1}]$. Argue that in this group we can take $n = s$, so A is an square matrix of order n .
5. Let A' be *any* submatrix of A of order $n - 1$. Justify, stating the relevant results, that $\widetilde{\Delta}_K := \det A'$ is an invariant of K.
6. Compute a couple of examples and convince yourself that $\Delta_K = \pm t^r \widetilde{\Delta}_K$ for some integer r.
7. In the course we defined $\mathcal{W}(D)$ for any tangle diagram D. What is the shape of the Alexander matrix for an XC-tangle? How does it behave under merging?

References:

- *Introduction to Knot Theory*, RH Crowell and RH Fox (chapter 7 and 8)
- *A quick trip through knot theory*, RH Fox.
- *An Introduction to Knot Theory*, WBR Lickorish, (page 117),
- Notes by Jorge (upon request)

5 The invariant Z_{Dlb}

In this talk, you will further investigate the invariant Z_{Dlb} for the Dilbert algebra. You will show that for a 0-writhe long knot K, we have $Z_{Dlb}(K) = \Delta_K(-1)$ (in the second homework set you were asked to show that $Z_{Dlb}(K)$ is an integer). Besides, $\mathbf{i}$ will denote the imaginary unit.

1. Show that the map
$$\varphi : Dlb \to \mathcal{M}_2(\mathbb{C}) \quad , \quad \varphi(l) := e_{11} \ , \ \varphi(b) := e_{12} \ , \ \varphi(d) := e_{21} \ , \ \varphi(a) := e_{22},$$
where e_{ij} denotes the elementary matrix with 1 in the (i, j) entry and 0 else defines an algebra isomorphism.

2. Let X_{ij}^{Dlb}, C_s^{Dlb} be the XC-algebra structure elements of Dlb as in Lecture 6. Show that
$$\varphi^{\otimes\{i,j\}}(X_{ij}^{Dlb}) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_j + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}_i \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}_j + 2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_j$$
and
$$\varphi^{\otimes\{s\}}(C_s^{Dlb}) = \mathbf{i}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_s$$

3. Let $\lambda \in \mathbb{C}$. Show that
$$X_{ij}(\lambda) := \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}_j + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}_i \begin{pmatrix}-\lambda^{-1} & 0\\ 0 & 1\end{pmatrix}_j + 2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_j$$
and $C_s := \varphi^{\otimes\{s\}}(C_s^{Dlb})$ define a XC-algebra structure on $\mathcal{M}_2(\mathbb{C})$ for every value of λ. It is enough that you show two out of the five defining axioms (of your choice!). (*Hint.* $X_{ij}^{-1}(\lambda) = X_{ij}(\lambda^{-1})$)

4. Let $A(\lambda)$ denote the algebra $\mathcal{M}_2(\mathbb{C})$ with the XC-algebra structure elements $X_{ij}(\lambda)$ and C_s. Show that for any long knot K, the value $Z_{A(\lambda)}(K)$ is independent of λ. (*Hint.* Use induction on the number of crossings and $X_{ij}^{-1}(\lambda) = X_{ij}(\lambda^{-1})$.) You are allowed to use this item without proof.

5. Consider the algebra $\mathcal{M}_2(\mathbb{C}[q, q^{-1}])$ (viewed as an algebra over $\mathbb{C}$). Show that the evaluation map $ev : \mathbb{C}[q, q^{-1}] \to \mathbb{C}$ given by $ev(f(q)) := f(e^{\pi\mathbf{i}/4})$ induces a $\mathbb{C}$-algebra map $ev : \mathcal{M}_2(\mathbb{C}[q, q^{-1}]) \to \mathcal{M}_2(\mathbb{C})$.

6. Endow $\mathcal{M}_2(\mathbb{C}[q, q^{-1}])$ with the XC-algebra structure[1] determined by the elements X_{ij}^{Alex}, C_s^{Alex} that recover the Alexander polynomial as in Lecture 8 (viewed with complex coefficients). Compute the elements $ev^{\otimes\{i,j\}}(X_{ij}^{Alex})$ and $ev^{\otimes\{s\}}(C_s^{Alex})$ in $\mathcal{M}_2(\mathbb{C})$. What is the value of the invariant $Z_{\mathcal{M}_2(\mathbb{C})}(K)$ of a long knot K for these structure elements?

[1] We consider $\mathcal{M}_2(\mathbb{C}[q, q^{-1}])$ instead of $\mathcal{M}_2(\mathbb{C}(q))$ so that the evaluation map is well-defined. This change has no effect on the XC-algebra structure.

7. Argue that the elements $X'_{ij} := \mathbf{i} \cdot ev^{\otimes\{i,j\}}(X^{Alex}_{ij})$ and $C'_s := ev^{\otimes\{s\}}(C^{Alex}_s)$ determine an XC-algebra structure on $\mathcal{M}_2(\mathbb{C})$. Check that $C_s = C'_s$. For what value of λ is $X_{ij}(\lambda) = X'_{ij}$?

8. Show that if $\mathcal{M}_2(\mathbb{C})$ is endowed with the XC-algebra structure determined by X'_{ij} and C'_s, then $Z_{\mathcal{M}_2(\mathbb{C})}(K) = \mathbf{i}^{w(K)}\Delta_K(-1)$, where $w(K)$ is the writhe of K.

9. Conclude that for a 0-writhe long knot K, we have $Z_{Dlb}(K) = \Delta_K(-1)$.

6 Representations of the braid group via the Yang-Baxter equation

In this project you will study a particular kind of representation of the braid group B_n that will lead to link invariants. In particular, you will revisit the Jones polynomial from this point of view.

1. Briefly explain the braid group B_n and state the Alexander and Markov theorems, and consider the standard presentation of B_n (Artin's presentation) with generators $\sigma_1, \dots, \sigma_{n-1}$.

2. Define an R-matrix $R : V \otimes V \to V \otimes V$ as a solution of the Yang-Baxter equation. Show that any R-matrix gives rise to a representation of B_n,

$$\rho_{R,n} = \rho_n : B_n \to \mathrm{End}(V^{\otimes n}), \qquad , \qquad \rho(\sigma_i) = id^{\otimes n-i} \otimes R \otimes id^{\otimes n-i-1}.$$

3. Show that

$$\varphi : \mathrm{End}(V) \otimes \mathrm{End}(V) \overset{\cong}{\to} \mathrm{End}(V \otimes V) \qquad , \qquad f \otimes g \mapsto f \otimes g$$

is a vector space isomorphism. What does $f \otimes g$ mean in each case? If we fix a basis for V, what is the corresponding isomorphism in terms of matrices?

4. Let V be a 2-dimensional vector space over $\mathbb{Q}(q)$ with a fixed choice of basis (e_1, e_2), and let $X \in \mathrm{End}(V) \otimes \mathrm{End}(V)$ be the element X that recovered the Jones polynomial in Lecture 7. Let $P : V \otimes V \to V \otimes V$ be the map $P(e_i \otimes e_j) := e_j \otimes e_i$. Write down the element $R := (P \circ \varphi)(X)$ and show that R is indeed an R-matrix.

5. Write $\rho_n : B_n \to \mathrm{End}(V^{\otimes n})$ for the braid group representation associated to R, and let $C \in \mathrm{End}(V)$ the C element from Lecture 7. We write $C^{\otimes n}$ for the corresponding map $C^{\otimes n} : V^{\otimes n} \to V^{\otimes n}$. Show that for a braid β,

$$tr(C^{\otimes n} \circ \rho_n(\beta)) \in \mathbb{Q}(q)$$

is invariant under the Markov moves.

6. Make an explicit computation in the case of the braid σ_1^2.

7. Argue that the value $tr(C^{\otimes n} \circ \rho_n(\beta))$ is exactly the same as the one that you would obtain if you compute the invariant $Z_{\mathrm{End}(V)}(cl(\beta))$ for the XC-algebra $\mathrm{End}(V)$. Conclude that

$$J_{cl(\beta)}(t) = tr(C^{\otimes n} \circ \rho_n(\beta)).$$

References:

- *Quantum Invariants, a study of knots, 3-manifolds and their sets,* T. Ohtsuki. (§2.2)

7 Slice and ribbon knots

In this project you will learn about slice and ribbon knots and the celebrated Slice-Ribbon Conjecture, a long-standing open problem in knot theory posed by Fox in the 1960s.

1. Define slice knots as those that bound a locally flat disc D^2 in $\mathbb{R}^4$. What does the *locally flat* condition mean? What happens if one removes this condition?
2. Explain how sliceness corresponds to a *movie* from the knots to the unknot. Give an example of a slice knot and show that it is indeed slice using these movies.
3. Show that for any oriented knot K, we have that $K\#(-\overline{K})$ is a slice knot.
4. State the fact that any slice knot has a block Seifert matrix with a zero block matrix in the top left block and use this to show the Fox-Milnor theorem: the Alexander polynomial of any slice knot K factors as $\Delta_K(t) = f(t)f(t^{-1})$ for some polynomial $f \in \mathbb{Z}[t]$.
5. Define ribbon knots. Show that any ribbon knot is slice. State the slice-ribbon conjecture.
6. Find a special type of Seifert surface for ribbon knots that produces a Seifert matrix of the form mentioned in part 4).
7. Suppose that K is a ribbon knot. How would you construct a ribbon disc for K using the merging and doubling operations of XC-tangles?

References:

- *Slice Knots: Knot Theory in the 4th Dimension*, P Teichner
- *An introduction to knot theory,* WBR Lickorish,
- Notes by Jorge (upon request)
- Alexander polynomial of ribbon knots, Sheng Bai, https://arxiv.org/abs/2103.07128

8 n-colourability

In this project you will generalise 3-colourability to n-colourability and study further properties of both.

1. Show that 3-colourability is a knot invariant.
2. Let $col_3(K)$ be the number of 3-colourings of a knot K. Show that this number is always a power of 3. Is $col_3(K) \in \mathbb{Z}$ also knot invariant?
3. Expand the previous definition to n-colourability. Adapt your previous proof to show that n-colourability is also a knot invariant.
4. In the third homework set you showed that the figure-of-eight knot has no non-trivial 3-colourings. Show that this knot has non-trivial 5 colourings.
5. Let $Col_n(K)$ be the set of n-colourings of a knot K. Endow this set with a structure of an abelian group.

6. Sketch a bijection
$$Col_n(K) \xrightarrow{\cong} \mathrm{Hom}_{MR}(\mathcal{W}(K), D_{2n})$$
where D_{2n} denotes the dihedral group and $\mathrm{Hom}_{MR}(\mathcal{W}(K), D_{2n})$ denotes the set of representations $\mathcal{W}(K) \to D_{2n}$ sending Meridians to Reflections.

7. Look up and define the determinant of a knot, noting its relation to the Alexander polynomial.

8. State and prove the theorem that a knot is n-colorable iff n is divisible by the determinant.

References:

- *3-coloring and other elementary invariants of knots*, Józef H. Przytycki.
- Why knot?, Dan Silver, https://www.southalabama.edu/mathstat/personal_pages/silver/why.knot.pdf
- Knot colorings and Goeritz matrices, Sudipta Kolay, https://arxiv.org/pdf/1910.08044.pdf

9 Whitehead doubles of knots

In this project you will study a class of knots called *Whitehead doubles*. This class is particularly suitable to be studied with the tools we developed in this course

1. For a knot K in $\mathbb{R}^3$, define the Whitehead double topologically as an instance of a satellite knot. What is the pattern? And the companion?

2. Argue that there are several types of Whitehead doubles, depending on the sign of the clasp and the number of twists involved. a positive one and a negative one, see p.2 of the 2nd reference. positively clasped Whitehead double $D_+(K, q)$

3. Construct a double-humped Seifert surface for $D_+(K, 0)$.

4. Use the item above to show that $\Delta_{D_+(K,0)}(t) = 1$ for any K.

5. Can $D_+(K, 0)$ possibly be then the trivial knot?

6. Use the XC-tangle tools developed in the course to express $D_+(K, 0)$ algebraically.

7. Tweak the previous definition to obtain $D_+(K, q)$ for $q \in \mathbb{Z}$, the positively clasped q-twisted Whitehead double. What is its Alexander polynomial?

8. What can we say about the Jones polynomial of an (un)twisted Whitehead double: can it possibly be trivial?

References:

- *Knots*, Burde Zieschang.
- Knot Floer homology of Whitehead doubles, M. Hedden, https://arxiv.org/pdf/math/0606094.pdf
- *An introduction to knot theory,* WBR Lickorish,

10 Two-bridge knots and computations in $\mathbb{D}$

Two bridge knots or rational knots are a particularly nice class of knots. For such knots computations of $Z_{\mathbb{D}}$ are much easier than usual even though the number of crossings appear high. Many of the simplest knots are two-bridge knots.

1. Find out what is the bridge number of a knot. What are the knots of bridge number 1?
2. Knots with bridge number two have been classified. Discuss this classification.
3. What is the Schubert normal form of a two bridge knot? Show how to draw some examples.
4. Find a knot that has bridge number 3.
5. Prove that any two bridge knot has a diagram that can be written as $m_r^{a,b}(D)$ where D is an XC-diagram with $\mathcal{L}(D) = \{a, b\}$ and D is OU (except it may include C's).
6. Explain why $Z_{\mathbb{D}}$ is rather easy to compute for two-bridge knots in the sense that the complicated multiplication procedure in $\mathbb{D}$ needs to be carried out only once.
7. Find an expression for the invariant of the trefoil knot in terms of a general choice of $\mathbb{O}$ and $\mathbb{U}$ and specialize it in the cases of the concrete Hopf algebra examples presented in the lectures.

References:

- *Knots*, Burde Zieschang.
- *An introduction to knot theory,* WBR Lickorish,

11 Existence of a ribbon element in a Drinfeld double

In the final lectures we constructed the algebra $\mathbb{D}$ from Hopf algebras $\mathbb{O}$ and $\mathbb{U}$. This construction is known in the literature as the Drinfeld double construction. What was left open in the lectures is whether or not the element C actually exists in $\mathbb{D}$. To obtain C we needed to extract or adjoin a certain square root. When does this element exist? And when does it need to be adjoined? In this more challenging project you try to read the original research paper by Kauffman and Radford that gives a neat criterion for the existence of C in terms om some properties of the algebra $\mathbb{U}$.

- *A Necessary and Sufficient Condition for a Finite-Dimensional Drinfeld Double to Be a Ribbon Hopf Algebra* Kauffman, Radford, Journal of Algebra 1993.

12 Introduction to Khovanov homology

In this project you will study a different perspective on knots and knot invariants through homological algebra that was given by M. Khovanov in 1999. Extending the Jones polynomial in an unexpected direction he opened up an entire subfield known as *categorified knot invariants.*

1. Let $\langle - \rangle$ denote the Kauffman bracket and define $\langle D \rangle' \in \mathbb{Z}[q, q^{-1}]$ by the rules
   (a) $\langle \times \rangle' = \langle) (\rangle' - q \langle \asymp \rangle'$
   (b) $\langle D \coprod \bigcirc \rangle' = (q + q^{-1}) \langle D \rangle'$

(c) $\langle \bigcirc \rangle' = 1$

Show that both brackets are equivalent in the sense that they are related by

$$\langle D \rangle = A^n \langle D \rangle' \qquad , \qquad A^2 = -q^{-1},$$

where n is the number of crossings of D.

2. Let $n = n_+ + n_-$, where $n_\pm$ denotes the number of positive/negative crossings of D. Then show that the Jones polynomial in terms of $\langle - \rangle'$ is given by

$$J_L(q) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle'$$

where D is a diagram of L.

3. What are the changes if (c) is replaced by $\langle \emptyset \rangle' = 1$?

4. Argue why the Jones polynomial can be computed following the procedure explained in page 3 of Bar-Natan's paper using a cube of complete smoothings. How is this related to the skein trees we have used through the course?

5. Define graded vector space, graded dimension, graded Euler characteristic, chain complex, degree shift operator and height shift operator.

6. Construct chain complex $\mathcal{C}(D)$ for every diagram D of a link D.

7. Let χ_q denote the graded Euler characteristic. Show that

$$\chi_q(\mathcal{C}(D)) = J_L(q).$$

8. Sketch the construction of the chain maps in $\mathcal{C}(D)$, and conclude stating Khovanov's theorem saying that the homology groups of this chain complex are link invariants.

References:

- D. Bar- Natan, *On Khovanov's categorification of the Jones polynomial*, 2002.

13 Knot theory computations with Mathematica

In this project you will give an overview of the Mathematica KnotTheory package available at the website of the knot atlas. You will also do some demonstrations on how it can be used to illustrate computations done in this course.

1. Download the KnotTheory package and browse the Manual on the knot atlas website.

2. Figure out how to get it to compute the Jones and Alexander polynomials of the all prime knots with less than 11 crossings. Are there any pairs of knots with same Jones but different Alexander? Which ones? And conversely, pairs with same Alexander and different Jones.

3. Find a way to also compute the Z_{Dlb} automatically on the same knot table using the conversion program Rot.nb from.

4. Can you also compute other invariants mentioned in this course using the Mathematica files one the course website?

References:

- The Knot atlas website, D. Bar-Natan, S. Morrisson, www.katlas.org
- Rot: a package for conversion to XC tangles, D. Bar-Natan drorbn.net/APAI

14 Universality of the braid category

The goal is to show that the braid category is the free strict braided category generated by one object.

1. Show for vector spaces, groups, algebras or your favourite algebraic structure the following universal property of the free construction: if $\bullet$ represents the singleton and $F(\bullet)$ is the free vector space/group/algebra/... on $\bullet$, then given a vector space/group/algebra/... M and $x \in M$, then there exists a unique vector space/group/algebra/... homomorphism $F(\bullet) \to M$. Argue that this amounts to saying that

   $$\mathrm{Hom}(F(\bullet), M) \cong \mathrm{Hom}_{Sets}(\bullet, M),$$

   where the left hand side Hom denotes vector space/group/algebra/... homomorphisms. Why should this be called "universal property"?

2. Recall what a category and a functor is. Give examples.

3. Define what a (strict) tensor category and a tensor functor are. Give examples. Is the category of vector spaces a strict tensor category?

4. Define what a (strict) braided tensor category and a braided functor is. Give examples. Is the category of vector spaces braided?

5. Describe the braid category $\mathcal{B}$ and exhibit in detail a structure of strict braided tensor category.

6. Show the following universal property of the braid category $\mathcal{B}$: if $\mathcal{C}$ is a strict braided tensor category and $X \in \mathcal{C}$, then there exists a unique braided functor $F_X : \mathcal{B} \to \mathcal{C}$ such that $F_X(1) = X$.

7. Let $*$ denote the one-object discrete category. Construct a canonical functor $i : * \to \mathcal{B}$. Argue that the above universal property is equivalent to the following: if $G : * \to \mathcal{C}$ is a functor, then there exists a unique braided functor $\widehat{G} : \mathcal{B} \to \mathcal{C}$ such that $\widehat{G} \circ i = G$. Why should this called "universal property" of $\mathcal{B}$?

8. (BONUS) Use the universal property above and a canonical isomorphism $\mathrm{End}(V)^{\otimes n} \cong \mathrm{End}(V^{\otimes n})$ to exhibit $Z_{\mathrm{End}(V)}$ on braids as a braided functor $\mathcal{B} \to \mathbf{Vect}$.

15 Your own suggestion goes here

If you prefer to propose your own topic that's perfectly fine as long as you tell us in advance so we can see if the topic is suitable.