Topics in Topology 2022

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These are the lecture notes for a Master's course on three and four-manifolds taught at University of Groningen during Winter 2022.

Contents

1 Lecture 1 (7/2)

The Poincaré Conjecture We start by motivating our study of 3- and 4-dimensional manifolds with the Poincaré conjecture. A closed manifold will mean a compact manifold without boundary.

Conjecture 1.1. *Suppose a closed* 3*-manifold* M *has trivial first homology. Then* $M \cong S^3$*.*

Poincaré himself proved this to be false, through the construction of the Poincaré homology sphere S_p. This object is constructed as follows.

1. Construct a dodecahedron as a CW complex.

2. Glue opposite faces with a twist of $5\pi/3$.

Why does this provide a counter example? Because $\pi_1(S_p) \cong \mathrm{BI}$, the binary icosahedral group, and the commutator subgroup of BI is itself, which implies that its abelianisation produces the trivial group. As we shall later see, the first homology group is isomorphic to the abelianisation of the fundamental group. But the fundamental group is a topological invariant, so S_p cannot be homeomorphic to S^3. In light of this realisation, Poincaré corrected his initial conjecture to the form in which we know it today, which was proven to be correct:

Theorem 1.1 (The Poincaré Conjecture)**.** *A closed* n*-manifold* M *is homotopy equivalent to* S^n *if and only if* $M \cong S^n$*.*

TBA: timeline of proofs of the Poincaré conjecture.

Using Knots to Study Manifolds

Definition 1.2. An isotopy on a manifold M is a continuous family of homeomorphisms from M to M. That is, a continuous map $h\colon [0,1] \times M \to M$ such that $h_t\colon M \to M$ is a homeomorphism for all $t \in [0,1]$.

In what is to follow, we will not distinguish between an embedding f, and its image in the target space.

Definition 1.3. A link is an equivalence class of smooth embeddings $f\colon \bigsqcup_{k=1}^{c} S^1 \to S^3$. The number c is called the number of link components. We declare an equivalence relation on the set of links by $f_K \sim f_L$ if and only if there exists an isotopy $h_t : S^3 \times I \to S^3$ such that the following diagram commutes:

$$\begin{array}{ccc} \bigsqcup S^1 & \xrightarrow{f_L} & S^3 \\ & \underset{f_K}{\searrow} & \downarrow h_t \\ & & S^3 \end{array}$$

Furthermore, we require that $h_0(K) = K$ and $h_1(K) = L$.

RvD: Where K and L are the images of f_K and f_L? Bram: Yes, I mentioned above that we don't distinguish between the maps and the images (for embeddings)

We want to use knots to encode 3- and 4-manifolds. Why knots? Attaching a disk $D^2 \to M$ means embedding the boundary $S^1 \to M$, which naturally produces a knot in M. This is one way in which knot theory may arise in the study of manifolds. Another way is by performing surgery on 3-manifolds. Let $K \subset M$ be a link. Let $T(K)$ be a tubular neighbourhood, which in this case means it is homeomorphic to a union of solid tori. Then we can glue back more solid tori, in a possibly different way – thus obtaining a new 3-manifold.

Theorem 1.4 (Lickorish–Wallace)**.** *Every closed* 3*-manifold is obtained by performing surgery on link complements in the manner described above, for* $M = S^3$*.*

Definition 1.5. There are three types of Reidemeister moves, namely:

1. Twist and untwist in either direction.
2. Move one loop completely over the other.
3. Move a string completely over or under a crossing.

The Reidemeister moves are illustrated in Figure 1. Note that in move 1 and 3 we skip over the cases with a different type of crossing (i.e. positive instead of negative or vice versa).

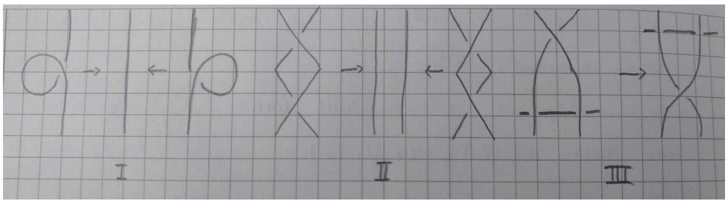


Figure 1: The three types of Reidemeister moves

Theorem 1.6 (Reidemeister)**.** *Two link diagrams L and L' define the same link class if and only if L and L' are connected by a sequence of Reidemeister moves.*

Definition 1.7. In an oriented link diagram, there are two types of crossings. We denote a positive crossing with a $+$-sign and a negative crossing with a $-$-sign. They are illustrated in Figure 2.

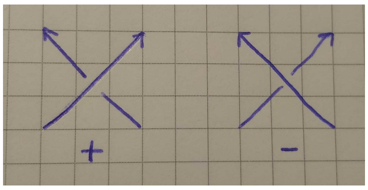


Figure 2: A positive crossing versus a negative crossing

So, if we cross over from left to right we have a positive crossing, but if we cross over from right to left we have a negative crossing.

Definition 1.8. Let L be an oriented link diagram with components $L_1, L_2, \ldots, L_n$. The linking number of two distinct components L_i, L_j is defined as

$$\mathrm{Lk}(L_i, L_j) = \frac{1}{2}\sum_{c} \mathrm{sign}(c),$$

where we run over all crossings c between L_i and L_j and $\mathrm{sign}(c)$ refers to the type of crossing as defined in Definition 1.7. For the entire link L, we define the linking number as the sum of the linking numbers of all pairs of components:

$$\mathrm{Lk}(L) = \sum_{1 \leqslant i < j \leqslant n} \mathrm{Lk}(L_i, L_j),$$

Ruben IJ: You multiply by 1/2 twice, so now every sign is being multiplied by 1/4. Is this correct? It seems a little strange.

2 Homology (Bram Brongers)

We will give a brief overview of (cellular) homology. After its definition, we shall review some basic facts, give geometric intuition, explain a few computational tools and give several examples.

The Definition of Cellular Homology Homology comes in different flavours. Usually, one is first introduced to singular homology. However, the singular homology groups of spaces are very hard to compute in general. That is why we want to develop cellular homology - because it provides a convenient setting for computations, when combined with some of the properties to be explained later on. Cellular homology is defined for CW complexes, so we will first state their definition as a reminder.

Definition 2.1. A **CW complex** is a a topological space X together with a sequence of subspaces $X_0 \subseteq X_1 \subseteq \dots$ such that $\cup_i X_i = X$, which satisfies the following.

1. We obtain X_n from X_{n-1} by attaching n-cells.
2. $U \subseteq X$ is open if and only if $U \cap X_n \subseteq X_n$ is open for each n.

The topological space X_n is called the n-skeleton of X.

Obtaining X_n from X_{n-1} by attaching n-cells means that there is a homeomorphism

$$X_n \cong X_{n-1} \cup_f J_n \times D^n$$

where J_n is an indexing set (possibly infinite) and $f : J_n \times \partial D^n \to X_{n-1}$ is an attaching map.

Remark 2.2. We recall that

$$X \cup_f Y := (X \bigsqcup Y)/\sim$$

where $f(a) \sim a$, for $f : A \to X$ with $A \subseteq Y$. In the present case, $A = J_n \times \partial D^n$, $X = X_{n-1}$ and $Y = J_n \times D^n$.

The cardinality of J_n is the number of n-cells in X. There is a category of CW complexes, in which we can consider two natural kinds morphisms. The first kind is just continuous maps $f : X \to Y$. The second kind is **cellular maps**, which are continuous maps between CW complexes such that $f(X_n) \subseteq Y_n$.

Remark 2.3. As is proved e.g. in the Mastermath Algebraic Topology 1 lecture notes, every continuous map between CW complexes is homotopic to a cellular map. Because homology will turn out to be homotopy invariant, we can always assume that we are dealing with a cellular map for the purposes of homology theory.

CW complexes are "nice" model spaces, in the sense that we can obtain information about e.g. manifolds from CW complexes. This is because, as mentioned previously, homology (and various other theories) are homotopy invariant, and every compact manifold is homotopy equivalent to a CW complex. Hence, we can transfer questions about the topology of these manifolds to CW complexes, where they are much easier to answer, due to convenient tools for computations such as cellular homology. To define it, we first need some more terminology.

Definition 2.4. A **chain complex** is a sequence of abelian groups

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

such that $\partial_k \circ \partial_{k+1} = 0$. We denote a chain complex by $(C_\bullet, \partial)$. Given a chain complex, we define its k-th **homology group** by

$$H_k(C_\bullet) = \frac{\ker \partial_k}{\text{im } \partial_{k+1}}$$

Exercise 2.5. Given a chain complex $(C_\bullet, \partial)$, show that $H_k(C_\bullet) = 0$ for all k if and only if the chain complex is exact. That is, $\ker \partial_n = \text{im } \partial_{n+1}$. Thus, the homology of a chain complex measures the obstruction to exactness.

For a CW complex, we define $C_n(X)$ to be the free abelian group on J_n generators, so each n-cell is a generator of $C_n(X)$. We want to turn this collection into a chain complex, so we can take its homology. Hence we need to construct a differential (or boundary operator). To this end, we make the following definition.

Definition 2.6. Let $f : S^n \to S^n$ be a smooth map. Then we define an integer number called the **degree** of f by

$$\int_{S^n} f^*\omega = \deg f \int_{S^n} \omega$$

where ω is the standard volume form on S^n. Note that we are implicitly assuming that the sphere is oriented, so that there is no ambiguity about the sign of the integral.

Remark 2.7. Proofs of these two facts below can be found in J. Lee's "Introduction to Smooth Manifolds".

1. By a theorem due to H. Whitney, every continuous map between smooth manifolds is homotopic to a smooth map. The usual definition of the degree of a map (in terms of singular homology) is a homotopy invariant, and therefore these definitions coincide (theorem 6.26).

2. It is not at all obvious why $\deg f$ is an integer (theorem 17.35). The usual definition through singular homology would illuminate this fact. We just accept it as a given. The cited proof uses de Rham cohomology. However, since de Rham cohomology is independent of what we develop here, there is no tautology.

Now we move on the definition of our boundary operator/differential, for the cellular homology of a CW complex X. Let $\alpha \in J_n$, $\beta \in J_{n-1}$ and let e_n^α be an n-cell, while e_{n-1}^β is an $(n-1)$-cell. We assume that all cells are oriented. Let f^α be the attaching map of of e_n^α. Then we can consider the sequence of maps

$$f^{\alpha\beta} : S^{n-1} \cong \partial e_n^\alpha \xrightarrow{f^\alpha} X_{n-1} \xrightarrow{\pi} X_{n-1}/(X_{n-1} \backslash e_{n-1}^\beta) \cong S^{n-1}$$

Here, f^α is the attaching map, and π is the quotient map which collapses everything except e_{n-1}^β to a point. Then $f^{\alpha\beta} : S^{n-1} \to S^{n-1}$ has a mapping degree, which is an integer. For the case $n = 2$, however, this is just the winding number of the map $f^{\alpha\beta} : S^1 \to S^1$.

Definition 2.8. Let $\alpha \in J_n$, so that e_n^α is one of the generators of $C_n(X)$. We define

$$\partial_n(e_n^\alpha) = \sum_{\beta \in J_{n-1}} \deg\,(f^{\alpha\beta}) e_{n-1}^\beta \in C_{n-1}(X)$$

This map $\partial_n : C_n(X) \to C_{n-1}(X)$ is extended by $\mathbb{Z}$-linearity to a group homomorphism.

Exercise 2.9. Work out concretely how to formulate the degree of the attaching map of a 1-cell. Start by giving the interval $[0, 1]$ an orientation.

The following proposition can be found in e.g. A. Hatcher's book (Cellular Boundary Formula on page 140).

Proposition 2.10. *The map $\partial_n : C_n(X) \to C_{n-1}(X)$ satisfies $\partial_n \circ \partial_{n+1} = 0$, and hence makes $(C_\bullet(X), \partial)$ into a chain complex.*

We remark that $C_\bullet : \mathrm{CW} \to \mathrm{Ch}_\mathbb{Z}$ is in fact a covariant functor from the category of CW complexes to the category of chain complexes of $\mathbb{Z}$-modules (i.e. abelian groups), if we restrict the morphisms between CW complexes to cellular maps.

Definition 2.11. Let $(C_\bullet, \partial)$ be the chain complex obtained from a CW complex X as above. Then we define the n-th homology group of X (with coefficients in $\mathbb{Z}$) by $H_n(X) := H_n(C_\bullet(X))$.

Suppose that $\dim X = n < \infty$. Then it follows immediately that $H_k(X) = 0$ for all $k > n$ and $k < 0$. We will denote $H_\bullet(X) := \oplus_{k=0} H_k(X)$ (also if $\dim X = \infty$).

Basic Properties We will now explain some of the basic properties of homology: induced maps, functoriality, homotopy invariance, excision, the long exact sequence and the Euler characteristic. Let e_n^α be an n-cell in X and h_{n-1}^β an $(n-1)$-cell in Y. Then we can consider the sequence of maps

$$S^{n-1} \cong \partial_n^\alpha \to X_{n-1} \xrightarrow{f} Y_{n-1} \to Y_{n-1}/(Y_{n-1} \backslash h_{n-1}^\beta) \cong S^{n-1}$$

Thus, the map f induces a map $f_* : H_\bullet(X) \to H_\bullet(Y)$, by the same construction we gave for the differential/boundary operator above, and we call f_* the **induced map** (with respect to f).

Theorem 2.12 (Functoriality of homology)**.** *The map $H_\bullet : CW \to GrAb$ is a covariant functor from the category of CW complexes to the category of (graded) abelian groups.*

What this tells us in more down-to-earth terms, is that the operation $X \mapsto H_\bullet(X)$ assigns a graded abelian group $H_\bullet(X)$ to each CW complex X, and a (graded) group homomorphism $f_* : H_\bullet(X) \to H_\bullet(Y)$ for every continuous map $f : X \to Y$, such that

1. $(g \circ f)_* = g_* \circ f_*$ for continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$.
2. $\mathrm{id}_* = \mathrm{id}$.

As alluded to previously, homology satisfies the following very convenient invariance property.

Theorem 2.13 (Homotopy Invariance of Homology)**.** *Let $f, g : X \to Y$ be continuous maps which are homotopic. Then $f_* = g_*$.*

This proof can be found in any textbook that touches upon algebraic topology. The following corollary is immediate, and is of crucial importance for many proofs and computations. We recall that two spaces X and Y are homotopy equivalent if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Corollary 2.14. *Homotopy equivalent spaces have isomorphic homology groups.*

To see the usefulness of this result in action, we will use it later to show that the dimension of a manifold is well-defined. Next, we want to discuss the long exact sequence of a pair of topological spaces (X, A), i.e. $A \subseteq X$. To do this, we will use a more general result.

Definition 2.15. Let $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ be chain complexes. Then a morphism of chain complexes $\varphi : C_\bullet \to C'_\bullet$ is a collection of group homomorphisms $\varphi_n : C_n \to C'_n$ which makes the following diagram commute:

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n \\ \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_n} \\ C'_{n+1} & \xrightarrow[\partial'_{n+1}]{} & C'_n \end{array}$$

With this definition in hand, it makes sense to talk about the kernels and images of morphisms of chain complexes (or **chain maps**), and in particular, to talk about exact sequences of chain complexes.

Theorem 2.16 (The Long Exact Sequence in Homology). *Let $(C_\bullet, \partial)$, $(C'_\bullet, \partial')$ and $(\overline{C}_\bullet, \overline{\partial})$ be chain complexes, and suppose that*

$$0 \to C'_\bullet \xrightarrow{\varphi} C_\bullet \xrightarrow{\psi} \overline{C}_\bullet \to 0$$

is an exact sequence of chain complexes. Then there is an exact sequence in homology

$$\dots \to H_{n+1}(C'_\bullet) \xrightarrow{\varphi_*} H_{n+1}(C_\bullet) \xrightarrow{\psi_*} H_{n+1}(\overline{C}_\bullet) \xrightarrow{\delta} H_n(C'_\bullet) \to \dots$$

The group homomorphism δ is called the connecting homomorphism.

The proof of this theorem is commonly referred to as a "routine exercise" in "diagram chasing", and it can be found in Serge Lang's "Algebra" (Theorem XX.2.1). We want to apply this to the situation mentioned above, i.e. that of a topological pair (X, A), where we assume that A is a subcomplex. We will refer to this as a **pair of CW complexes**. In this situation, we get a short exact sequence of chain complexes

$$0 \to C_\bullet(A) \xrightarrow{\iota} C_\bullet(X) \xrightarrow{\pi} C_\bullet(X)/C_\bullet(A) \to 0$$

Definition 2.17. The homology of the complex $C_\bullet(X)/C_\bullet(A) := C_\bullet(X, A)$ is called the **homology of X relative to A**, or **relative homology** for short. It is denoted by $H_\bullet(X, A)$.

The relative homology tells us precisely how much of the homology of X, is captured by the homology of A. Indeed, $H_\bullet(X, A) = 0 \iff H_\bullet(A) \cong H_\bullet(X) \iff \iota_*$ is an isomorphism, which will be exemplified by the next corollary.

Corollary 2.18. *Let (X, A) be a pair of CW complexes. Then the relative homology $H_\bullet(X, A)$ naturally fits into an exact sequence*

$$\dots \to H_{n+1}(A) \xrightarrow{\iota_*} H_{n+1}(X) \xrightarrow{\pi_*} H_{n+1}(X, A) \xrightarrow{\delta} H_n(A) \to \dots$$

Next, we wish to discuss an incredibly useful tool, known as the Mayer-Vietoris sequence.

Theorem 2.19 (Mayer-Vietoris). *Let $U, V \subseteq X$ be subcomplexes such that $U \cup V = X$ and $U \cap V$ is a subcomplex. Then the following sequence is exact:*

$$\dots \to H_n(U \cap V) \xrightarrow{\iota_*^U \oplus \iota_*^V} H_n(U) \oplus H_n(V) \xrightarrow{i_*^U - i_*^V} H_n(X) \xrightarrow{\delta} H_{n-1}(U \cap V) \to \dots$$

where ι and i denote the sensible inclusion maps, and δ is the connecting homomorphism.

Remark 2.20. The connecting homomorphism shows up again above, because of the fact that the Mayer-Vietoris sequence is induced by an exact sequence of chain complexes. However, the exact sequence in question is not

$$0 \to C_\bullet(U \cap V) \to C_\bullet(U) \oplus C_\bullet(V) \to C_\bullet(X) \to 0$$

See the Mastermath Algebraic Topology 1 notes on "The Small Simplices Theorem" for details.

Exercise 2.21. Give the n-sphere a suitable CW structure, and use it to compute $H_k(S^n)$ for all $k, n \geqslant 0$ by induction on n and the Mayer-Vietoris sequence.

The final theorem we want to include, is the excision theorem. The tools used in the proof of this theorem are somewhat beyond the scope of a brief explanation, so we just cite it and refer to A. Hatcher's book (Theorem 2.20).

Theorem 2.22 (The Excision Theorem)**.** *Let $U \subset A \subseteq X$ such that (X, A) is a pair of CW complexes, and $\overline{U} \subseteq int(A)$. Then there is an isomorphism*

$$H_\bullet(X, A) \cong H_\bullet(X \backslash U, A \backslash U)$$

Remark 2.23. For this to be well-defined in the context of cellular homology, we of course need $(X \backslash U, A \backslash U)$ to be a pair of CW complexes. This is quite restrictive, and is not necessary in the context of singular homology, where the theorem also holds.

You will have the opportunity to use these tools in the following computations.

Exercise 2.24. Give the torus $T^2 := S^1 \times S^1$ a CW structure and compute its homology. What about the solid torus $\mathbb{T}^2$? Hint: for the latter, use homotopy invariance.

Exercise 2.25 (Homology of Knot Complement)**.** Let $K : S^1 \to S^3$ be a knot, and let $T(K)$ be a tubular neighbourhood. Then we can pick $T(K) \cong \mathbb{T}^2$. Let $M = S^3 \backslash T(K)$.

1. Find a suitable cover for M which makes it convenient to use the Mayer-Vietoris theorem for the computation of $H_\bullet(M)$.
2. Assume that the connecting homomorphism $H_3(S^3) \cong \mathbb{Z} \xrightarrow{\delta} H_2(T^2) \cong \mathbb{Z}$ is an isomorphism. Use this to calculate $H_\bullet(M)$.

We will end this section with the definition of an important concept, namely the Euler characteristic. Suppose that $\dim X = n$. We know frow from linear algebraic considerations, namely the rank-nullity theorem, that the following holds:

$$\sum_{k=0}^{n} (-1)^k \text{rank } H_k(X) = \sum_{k=0}^{n} (-1)^k \text{rank } C_k(X) = \sum_{k=0}^{n} (-1)^k |J_k|$$

Exercise 2.26. Prove the first equality above.

Definition 2.27. The Euler characteristic $\chi(X)$ of a topological space X is defined by

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \text{rank } H_k(X)$$

Hence, for a CW complex, the Euler characteristic coincides with the alternating sum of the k-cells that are used to construct it. As we will see later in the course, the Euler characteristic completely characterises 2-dimensional compact orientable manifolds up to homeomorphism.

Geometric Intuition and Examples We have now given the definition of cellular homology, and several of the basic properties and computational tools. Let us now indicate why we should care enough about homology, to go through the trouble of developing this machinery and computing it. For the concerned reader: we will give the intuition that is based on singular homology, and it may not necessarily be apparent that cellular homology is related to this. However, the two homologies are isomorphic, if we take singular homology with integer coefficients, so this is a perfectly valid thing to do. For those not familiar with singular homology, ignore the remark above.

Proposition 2.28. *Suppose that the number of components of X is m. Then $H_0(X) \cong \mathbb{Z}^m$.*

For the case $n = 1$, we have a similarly intuitive and useful result. Let $[G, G]$ denote the commutator subgroup of a group G.

Proposition 2.29. *Let $\pi_1(X)$ be the fundamental group of X, and assume X to be path-connected. Then*

$$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)]$$

That is, $H_1(X)$ is the abelianisation of the fundamental group $\pi_1(X)$.

The proof of the proposition can be found e.g. in J. Lee's "Introduction to Topological Manifolds" (Theorem 13.14). The condition of path-connectedness will, in practice, not add any difficulties. This is because we shall investigate manifolds, for which connectedness implies path-connectedness. When studying manifolds, we can always look at the individual components in isolation.

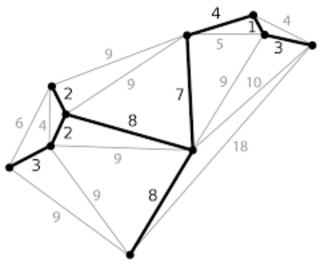
Now, let us recall some more terminology from group theory, namely the **rank** of an abelian group G. Let G_{tor} denote the torsion subgroup. Then $G/G_{\text{tor}} \cong \mathbb{Z}^k$ for some integer k, called the rank of G.

Proposition 2.30. *Let $n > 0$. The rank of $H_n(X)$ is the number of n-dimensional "holes" in X.*

The notion of an n-dimensional hole is not a "formal" one, and this is just to give some intuition about homology.

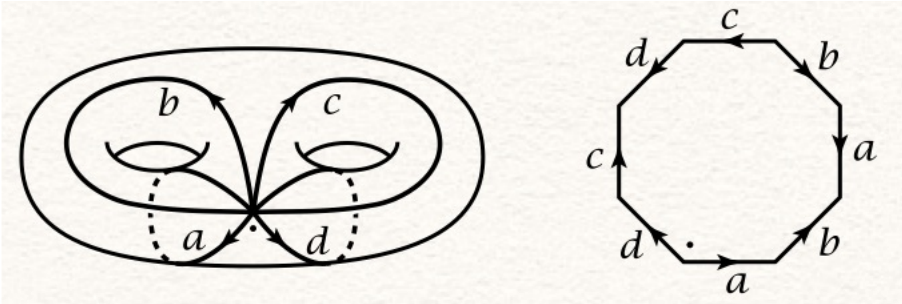
Examples 2.31. Let us explain what we mean by the proposition above in dimensions we can actually visualise.

1. A (simple undirected and connceted) graph can be given a natural CW complex structure, taking a 0-cell for every vertex, and a 1-cell for every edge. Then $H_1(X) \cong \mathbb{Z}^m$, where m is the number of cycles in the graph X. The way to obtain this result, is to use a spanning tree T, i.e. a subgraph which is a tree (contains no cycles) and contains every vertex of the original graph. An example is given below.



   Denote the set of edges in X by E, and the set of edges in T by E_T. Let $\Lambda = E \backslash E_T$. Then $H_1(X) \cong \mathbb{Z}[\Lambda]$. Geometrically, this means that each edge that is left out of the spanning tree, gives us a non-contractible loop which represents a generator of the first homology $H_1(X)$.

2. Let $X = \Sigma_g$ be the compact orientable surface of genus g (which is unique up to homeomorphism, as we shall see later in the course). Then $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$. Generators are given by representatives of the homotopy classes of non-contractible loops in Σ_g. To see this, consider the follow CW structure (for the case $g = 2$).



In much the same fashion, we can find a polygonal representation for arbitrary Σ_g, we merely need to take the regular $4g$-sided polygon and identify the edges in a suitable manner. As can be seen above on the right, the attachment map runs through each edge twice, in opposite orientation. Hence, the winding number of the attaching map for each circle (in the figure on the left) is $1 - 1 = 0$. It follows that deg $f^{\alpha\beta} = 0$ for all $\beta \in J_1$. Consequently, the cellular chain complex is:

$$0 \to \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}^{2g} \xrightarrow{\cdot 0} \mathbb{Z} \to 0$$

From which we easily deduce that

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } k = 1 \\ 0 & \text{else} \end{cases}$$

3. Finally, and more generally, we have

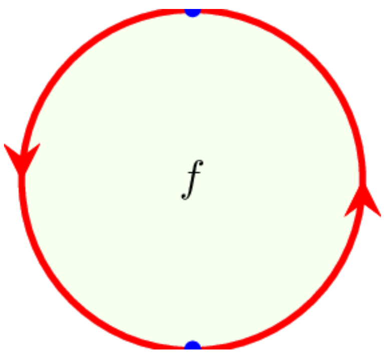
$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, n \\ 0 & \text{else} \end{cases}$$

which we can also reasonably interpret as "the n-dimensional sphere has one n-dimensional hole, and no other holes". This result is an easy consequence of our definition of homology, as the chain complex looks like

$$0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to 0$$

Namely, we can use one 0-cell, and one n-cell.

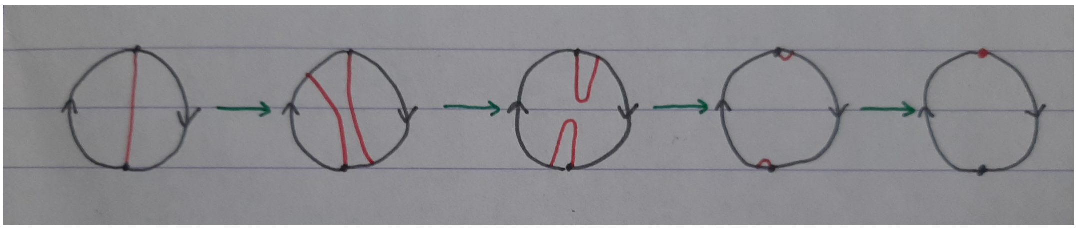
But what about the torsion subgroup? This is where matters become a bit more subtle, and goes to show why homology with integer coefficients gives more information than homology with coefficients in e.g. the real numbers. To gain some understanding of the meaning of the torsion part of a homology group, we will look at the projective plane $\mathbb{R}P^2$. We give it the following CW structure:



This CW structure consists of one 0-cell, one 1-cell and one 2-cell. First form a circle, then attach the disk with a degree 2 map. Note the map has degree 2 because we identify opposite points on the boundary of the disk. The cellular chain complex becomes

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \to 0 \implies H_k(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k = 1 \\ 0 & \text{else} \end{cases}$$

What is the significance of the torsion in $H_1(\mathbb{R}P^2)$? It means the following (see the illustration below). Picture $\mathbb{R}P^2$ as being the disk with antipodal points on the boundary identified. Drawing a straight line through the origin gives us a loop. This loop is not contractible. Indeed, we would have to make the ends of the line, on the edge of the disk, meet. But this is not possible, because every time we perturb one end, the opposite ends moves accordingly, to remain directly opposite. However, if we concatenate this loop with itself, we **do** get a contractible loop. This is precisely why we get the integers modulo two, rather than some other integer. Generalising this intuition to higher dimensions may be somewhat challenging. Below is an illustration of the procedure used to contract the concatenated path to a loop. Note that in the first step of the procedure, we actually have **two** lines, since we concatenated the line with itself.



Exercise 2.32. Compute the homology of complex projective space $\mathbb{C}P^1$, and use this intuition to compute the homology of $\mathbb{C}P^n$.

As a final note, we want to mention some applications to manifolds, since they are to be the primary subject of this course. Suppose that $\dim M = n$ for some manifold M that we are interested in. We will assume that M is compact. Then the following proposition tells us that homology gives information about the orientability of M.

Proposition 2.33. *$H_n(M) = 0 \iff M$ is non-orientable. Conversely, $H_n(M) \cong \mathbb{Z} \iff M$ is orientable.*

As we know, $\mathbb{R}P^2$ is indeed non-orientable, which is reflected by the homology computation above, which revealed that $H_2(\mathbb{R}P^2) = 0$, whereas $H_2(\mathbb{C}P^1) \cong \mathbb{Z}$, so complex projective space is orientable. Finally, we will prove that the dimension of a manifold is well-defined.

Theorem 2.34. *Suppose that $\mathbb{R}^n \cong \mathbb{R}^m$. Then $m = n$.*

Proof. We use the fact that homology is homotopy invariant, and that $\mathbb{R}^n \backslash \{0\} \simeq S^{n-1}$. Suppose without loss of generality that $n > m$ and that $\mathbb{R}^n \cong \mathbb{R}^m$. Then also $\mathbb{R}^n \backslash \{0\} \cong \mathbb{R}^m \backslash \{0\}$. Consequently, we find $H_\bullet(\mathbb{R}^n \backslash \{0\}) \cong H_\bullet(\mathbb{R}^m \backslash \{0\})$. By homotopy invariance, this implies $H_\bullet(S^{n-1}) \cong H_\bullet(S^{m-1})$. However, we know that $H_{n-1}(S^{n-1}) = \mathbb{Z}$, while $H_{n-1}(S^{m-1}) = 0$. Contradiction. So $m = n$. Thus, the dimension of a manifold is well-defined. □

3 Homotopy groups (Ruben IJpma)

Recall that two maps $f, g : X \to Y$ of topological spaces are called *homotopic relative to* $A \subseteq X$ if there exists a continuous family $F_t : X \to Y$, where $t \in [0,1]$,[1] with the two properties (i) $F_0 = f$ and $F_1 = g$, (ii) $F_t|_A = f|_A = g|_A$ for all t. The family F_t is called a *homotopy* from f to g relative to A, and we write $f \simeq_A g$. The relation $\simeq_A$ is an equivalence relation. The set of *homotopy classes* relative to A is denoted as

$$[X,Y]_A := \{\text{continuous maps } f : X \to Y\}/\simeq_A .$$

Further recall that if $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ and $Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y$ are CW-structures, then a continuous map $f : X \to Y$ is called *cellular* if $f(X_n) \subseteq Y_n$ for all n. Moreover, $A \subseteq X$ is called a *subcomplex* of X if $X_{n+1} - (A \cap X_n)$ is a union of open $(n+1)$-cells, for all n.

Theorem 3.1 (Cellular approximation theorem)**.** *Let X and Y be CW-complexes as above, and let some n-skeleton $A \subseteq X$ be a subcomplex. If $f : X \to Y$ is continuous and $f|_A$ is cellular, then f is homotopic relative to A to a cellular map.*

J: A is just a subcomplex right? (not sure what the n-skeleton part means)

Definition 3.2. Fix base points $s_0 \in S^n$ and $x_0 \in X_0$. Then denote

$$\pi_n(X, x_0) := [(S^n, s_0), (X, x_0)]_{s_0}.$$

For $n = 1$, this set coincides with the fundamental group of X at x_0. For $n \geqslant 2$, we will show that $\pi_n(X, x_0)$ likewise admits a group structure. In contrast to the fundamental group, this group structure will always be commutative.

Actually, it is already possible to compute $\pi_n(S^m, x_0)$ for $n < m$. For this, we need to consider maps $f : (S^n, s_0) \to (S^m, x_0)$. We view the spaces S^n, S^m as CW-complexes X, Y as follows:

$$X_i = \{s_0\} \text{ for } 0 \leqslant i < n, X_i = S^n \text{ for } i \geqslant n,$$
$$Y_i = \{x_0\} \text{ for } 0 \leqslant i < m, Y_i = S^m \text{ for } i \geqslant m.$$

The restriction $f|_{X_0}$ is certainly cellular, so f is homotopic relative to $\{s_0\}$ to a cellular map g, by the cellular approximation theorem. It follows that $g(S^n) = g(X_n) \subseteq Y_n = \{x_0\}$. So, f is homotopic to the constant map $s \mapsto x_0$. This argument shows that $\pi_n(S^m, x_0)$ consists only of a single element. When $n \geqslant m$, the situation is vastly different. We shall return to the special case $m = n$ later.

Exercise 3.3. Show that for $n = 0$, the elements of $\pi_0(X, x_0)$ can be viewed as the path components of X.

To define the group operation on $\pi_n(X, x_0)$, let $I^n = [0,1]^n$ be the n-dimensional cube. Appropriately identifying S^n with $I^n/\partial I^n$ allows us to identify the elements of $\pi_n(X, x_0)$ with the elements of

$$[(I^n, \partial I^n), (X, x_0)]_{\partial I^n}.$$

For $1 \leqslant i \leqslant n$, define the $*_i$-product of two 'balloons' $\alpha, \beta : (I^n, \partial I^n) \to (X, x_0)$ by

$$(\alpha *_i \beta)(t_1, \ldots, t_n) := \begin{cases} \alpha(t_1, \ldots, t_{i-1}, 2t_i, t_{i+1}, \ldots, t_n) & \text{if } 0 \leqslant t_i \leqslant \frac{1}{2}, \\ \beta(t_1, \ldots, t_{i-1}, 2t_i - 1, t_{i+1}, \ldots, t_n) & \text{if } \frac{1}{2} \leqslant t_i \leqslant 1. \end{cases}$$

Moreover, set $[\alpha] *_i [\beta] := [\alpha *_i \beta]$. We want to show that $*_i$ is a well-defined operation, with identity being the homotopy class of the constant map sending I^n to x_0, and inverses $[\overline{\alpha}^i]$ defined by $\overline{\alpha}^i(t_1, \ldots, t_n) = \alpha(t_1, \ldots, t_{i-1}, 1 - t_i, t_{i+1}, \ldots, t_n)$. Since only the ith coordinate is involved in all of the expressions, the proof of these facts is the same as the proof for the existence of the fundamental group.

[1] Here, *continuous family* means that the map $X \times [0,1] \to Y$ given by $(x,t) \mapsto F_t(x)$ is continuous.

Exercise 3.4. Find a quotient map $q : I^n \to S^n$ such that the only non-trivial fibre is $q^{-1}(s_0) = \partial I^n$.

Lemma 3.5 (Eckmann-Hilton). *Let M be a set with two binary operations $*$ and $\circ$ that both admit a two-sided unit element and satisfy $(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$ for all $a, b, c, d \in M$. Then the two operations coincide, and they are associative and commutative.*

It is easy to check that in fact $(\alpha *_i \beta) *_j (\gamma *_i \delta) = (\alpha *_j \gamma) *_i (\beta *_j \delta)$ for all i, j (without taking homotopy classes). Also, we argued a little earlier that a unit element exists for the homotopy classes. Therefore, all $*_i$-products are identical, and they are associative and commutative. This finishes the construction of the group operation on $\pi_n(X, x_0)$.

Exercise 3.6. Prove Lemma 3.5.

The nth homotopy group of a space X depends in general on its basepoint x_0. Since any map $(S^n, s_0) \to (X, x_0)$ must have image lying in the path component of x_0, the entire group is determined by the path component of x_0. Furthermore, if γ is a path in X from y_0 to x_0, then there is an induced isomorphism $\gamma_* : \pi_n(X, x_0) \to \pi_n(X, y_0)$.

In contrast, the nth *homology* group of a CW-complex X depends on all path components. Still, the homotopy groups can be related to the homology groups by what is known as the *Hurewicz map*, described as follows. By definition of cellular homology, the group $H_n(S^n; \mathbb{Z})$ is generated by $[e_n]$, where e_n is a single n-cell. Now, if $\alpha : (S^n, s_0) \to (X, x_0)$ is any map, then we consider the induced map $\alpha_* : H_\bullet(S^n; \mathbb{Z}) \to H_\bullet(X; \mathbb{Z})$ described earlier. From here, we have a well-defined map, called the *Hurewicz map*

$$\begin{aligned} h_n : \pi_n(X, x_0) &\to H_n(X; \mathbb{Z}) \\ [\alpha] &\mapsto \alpha_*([e_n]). \end{aligned}$$

This map is always a group homomorphism. If X is path connected, then h_1 is surjective, and its kernel equals the commutator subgroup of the fundamental group. (See Proposition 2.29.) For $n \geqslant 2$, we have

Theorem 3.7 (Hurewicz theorem (absolute version)). *Suppose that $\pi_i(X, x_0) = 0$ for $i < n$. Then the Hurewicz map $h_n : \pi_n(X, x_0) \to H_n(X; \mathbb{Z})$ is a group isomorphism.*

Corollary 3.8. *For $n \geqslant 2$, we have $\pi_n(S^n, s_0) \cong \mathbb{Z}$.*

Exercise 3.9. Check that the map h_n is a well-defined group homomorphism. (Hint: for well-definedness, use the homotopy invariance of homology.)

J: This seems pretty hard to be an exercise (!)

We finish our introduction of homotopy groups with the *Van Kampen theorem*, which is a tool to compute the fundamental group of many examples of spaces. The fundamental group often occurs as a free product of multiple groups. For groups $(G_1, e_1), (G_2, e_2)$, define their *free product* $G_1 * G_2$ as follows. Assume that $G_1 \cap G_2 = \varnothing$, otherwise consider $G_j \times \{j\}$ with appropriately modified operations. With $G_j^\bullet = G_j - \{e_j\}$, set

J: Unless G_1, G_2 are subgroups of another group, it doesn't make sense to consider their intersection. The union you use later is the disjoint union.

$$G_1 * G_2 = \bigcup_{n=0}^{\infty} \{f : \{1, \ldots, n\} \to G_1^\bullet \cup G_2^\bullet \mid \text{for all } i, f(i) \in G_j \implies f(i+1) \notin G_j\}.$$

Let $(f_1, \ldots, f_n), (g_1, \ldots, g_m) \in G_1 * G_2$, then the following defines a group structure on $G_1 * G_2$:

$$\begin{aligned} f * g &= (f_1, \ldots, f_{n-1}, f_n g_1, g_2, \ldots, g_m) && \text{if } f_n, g_1 \in G_j, f_n g_1 \neq e_j, \\ f * g &= (f_1, \ldots, f_{n-1}, g_2, \ldots, g_m) && \text{if } f_n, g_1 \in G_j, f_n g_1 = e_j, \\ f * g &= (f_1, \ldots, f_n, g_1, \ldots, g_m) && \text{else.} \end{aligned}$$

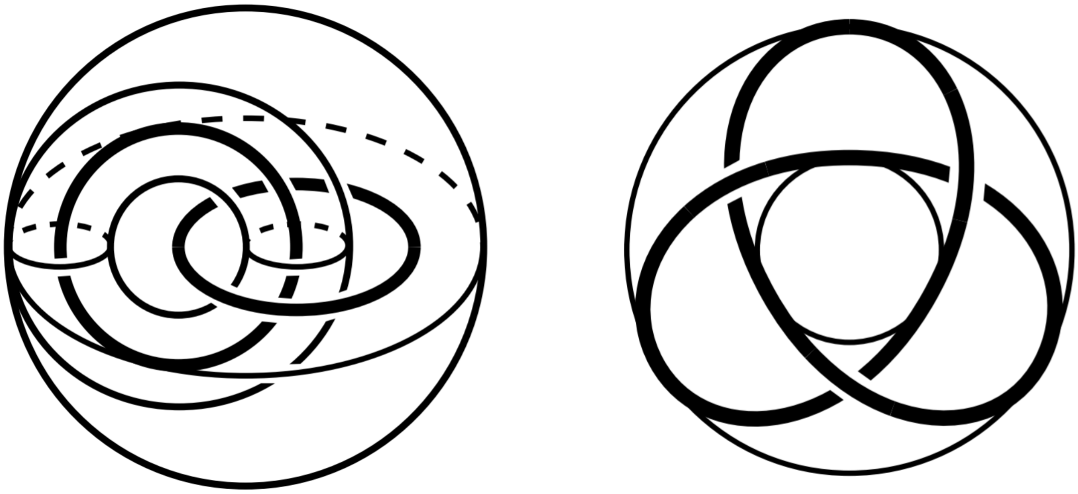


Figure 3: Right: $\mathbb{R}^3$ deformation retracts onto $S^2 \vee (S^1 \times S^1)$. Left: image of T in $S^1 \times S^1$. (Credit: Hatcher.)

Taking free products of multiple groups is associative and commutative, allowing for the likes of $G_1 * \cdots * G_n$. Free products have the property that when $\phi_i : G_i \to H$ are any homomorphisms, there exists a unique extension homomorphism $\Phi : G_1 * \cdots * G_n \to H$ with $\Phi((g_i)) = \phi_i(g_i)$ for all $g_i \in G_i$.

J: $f_i = f(i)$? I suppose the n-tuple is the image of f.

Furthermore, we will make use of the following notation. For a map $f : (X, x_0) \to (Y, y_0)$, there is a well-defined *induced* homomorphism of homotopy groups

$$\begin{aligned} f_* : \pi_n(X, x_0) &\to \pi_n(Y, y_0) \\ [\omega] &\mapsto [f \circ \omega]. \end{aligned}$$

Suppose that X is the union of path-connected open subsets $U_1, \ldots, U_n$, all containing the same basepoint x_0. Let $j^\alpha : U_\alpha \to X$ and $i^{\alpha\beta} : U_\alpha \cap U_\beta \to U_\alpha$ denote the inclusions, $\alpha, \beta = 1, \ldots, n$. For the induced homomorphisms $j^\alpha_* : \pi_1(U_\alpha, x_0) \to \pi_1(X, x_0)$, we consider the unique extension $\Phi : \pi_1(U_1, x_0) * \cdots * \pi_1(U_n, x_0) \to \pi_1(X, x_0)$. In the free product, we will consider elements of the form $(i^{\alpha\beta}_*(\omega), i^{\beta\alpha}_*(\omega)^{-1})$, for $\omega \in \pi_1(U_\alpha \cap U_\beta, x_0)$. That is, words consisting of $[\omega]$ as a homotopy class in U^α, followed by $[\omega^{-1}]$ as a homotopy class in U^β.

Theorem 3.10 (Van Kampen)**.** *With the hypothesis stated above, Φ is a surjective homomorphism. If moreover all triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ are path connected, then the kernel of Φ is generated by all elements of the form $(i^{\alpha\beta}_*(\omega), i^{\beta\alpha}_*(\omega)^{-1})$.*

J: I would suggest to restate the theorem only for the case where your space can be covered by two open subsets. Can you write a formula for the fundamental group of the total space in terms of generators and relations?

Examples 3.11.

1. A classical instance of the Van Kampen theorem is the *bouquet of circles* $X = \bigvee_{i=1}^n S^1$ with their common intersection point x_0 as the basepoint. Since every circle has a neighbourhood of x_0 which deformation retracts onto x_0, the Van Kampen theorem implies that $\pi_1(X, x_0)$ is isomorphic to the n-fold free product $\mathbb{Z} * \cdots * \mathbb{Z}$.

2. Suppose A consists of two linked circles in $\mathbb{R}^3$. Of interest is the fundamental group $\pi = \pi_1(\mathbb{R}^3 - A)$. One can figure out that $\mathbb{R}^3 - A$ deformation retracts onto $S^2 \vee (S^1 \times S^1)$, as seen in Figure 3. The Van Kampen theorem then implies that $\pi \cong \pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

3. The simplest non-trivial knot T can be seen as the image of $S^1 \times S^1$ defined as $z \mapsto (z^2, z^3)$, see Figure 3. In general, when embedding images $T_{m,n}$ of $z \mapsto (z^m, z^n)$ into $\mathbb{R}^3$, one can can find suitable deformation retracts leading to $\pi_1(\mathbb{R}^3 - T_{m,n}) \cong \mathbb{Z}/m * \mathbb{Z}/n$ by Van Kampen. Thus, $\pi_1(\mathbb{R}^3 - T) \cong \mathbb{Z}/2 * \mathbb{Z}/3$.

4 Lecture 2 (14/2): Handle decompositions

Definition 4.1. A topological space $\mathcal{M}$ is a manifold of dimension n if

- $\mathcal{M}$ is Hausdorff;
- $\mathcal{M}$ is second countable;
- $\mathcal{M}$ is locally Euclidean of dimension n.

Definition 4.2. A smooth manifold is a topological manifold M together with a choice of maximal atlas.

Now, a natural question is in which situations are a topological manifold and a smooth manifold the 'same'? That means, what can we say about mappings going from the space of smooth manifolds to the space of topological manifolds? It turns out[2] that, in general, such maps are not injective, nor surjective. Nevertheless, when the dimension $n = 0, 1, 2, 3$ of the manifolds we are considering, such a map must be a bijection[3].

From now on, we assume all manifolds to be smooth.

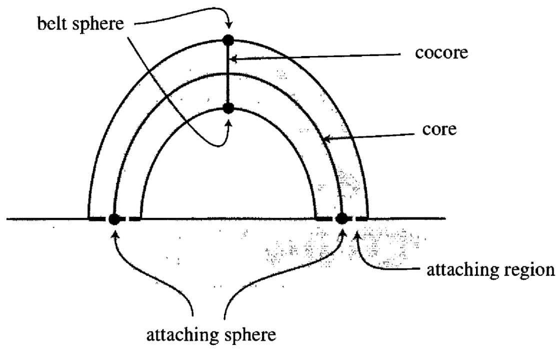
Definition 4.3. Let $0 \leqslant k \leqslant n$ and let $\mathcal{M}$ be a manifold with boundary, and where $\dim(\mathcal{M}) = n$. An n-dimensional k-handle h_k is a copy of $D^k \times D^{n-k}$ attached to $\partial\mathcal{M}$ along an embedding $\varphi : \partial D^k \times D^{n-k} \hookrightarrow \partial\mathcal{M}$.

Here, it is worth noticing that one attaches a $k-1$-handle h_{k-1} to k-handle h_k by an embedding φ, we have that $\varphi : \partial D^k \times D^{n-k} \hookrightarrow \partial\left(D^k \times D^{n-k}\right) = \left(\partial D^k \times D^{n-k}\right) \cup \left(D^k \times \partial D^{n-k}\right)$.

Example 4.4. Let's look at the case where $n = 3$ and $k = 1$. Hence, we are attaching a 1-handle to a 3-dimensional manifold $\mathcal{M}$. Here, φ will attach $\partial D^1 \times D^2 = \{0, 1\} \times D^2$ to $\partial\mathcal{M}$. Here, 'attaching' means introducing the equivalence relation $x \sim \varphi(x)$.

Example 4.5. For the case where $n = 3$ and $k = 2$, we are attaching a 2-handle – which is given by $D^2 \times D^1$; a coin! – to a manifold. The part of the 2-handle we are attaching to the manifold will be $\left(\partial D^2\right) \times D^1 = S^1 \times D^1$ (the outside strip of the coin).

Now, we will direct ourselves to some terminology that may turn out to be handy: the core, cocore and belt sphere of a handle, which are all subsets of the handle. The core is given by $D^k \times 0$, the cocore is $0 \times D^{n-k}$, which has the belt sphere $0 \times \partial D^{n-k}$ as its boundary.



Remark 4.6. It is true that when attaching a handle in the way we showed above, corners will not be smooth. Nevertheless, more advanced methods easily can rid of this non-smoothness obtained from gluing.

Definition 4.7. Let $f_0, f_1 : Y \hookrightarrow X$ be embeddings. f_0, f_1 are called *isotopic* if there exists an isotopy map $H : Y \times I \to X$ (where $I = [0, 1]$ an interval) such that $H_t := H(-, t)$, for $t \in I$, satisfying $H_0 = f_0$ and $H_1 = f_1$.

[2]This are two results proved by Milnor in the 50's and by Freedman in the 80's, respectively.
[3]Result proved by Edwin Moise.

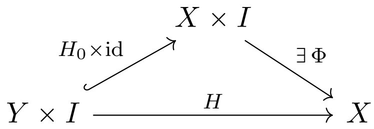
Theorem 4.8. *Isotopic embeddings $f_0, f_1 : (\partial D^k) \times D^{n-k} \hookrightarrow \partial\mathcal{M}$ give rise to diffeomorphic manifolds*

$$\mathcal{M} \cup_{f_0} h_k \cong \mathcal{M} \cup_{f_1} h_k.$$

Lemma 4.9 (Isotopy extension lemma)**.** *If Y is an compact (sub)manifold, then any isotopy $H : Y \times I \to X \subset \mathcal{M}$ (where again, $I = [0,1]$ an interval) can be extended to an ambient isotopy through diffeomorphisms*

$$\Phi : X \times I \to X, \qquad \textit{such that} \quad \Phi_0 = id, \quad H_t = \Phi_t,$$

and where Φ_t is a diffeomorphism for all $t \in I$.



Proof. According to the above lemma, there exists

$$\Phi : \partial\mathcal{M} \times I \to \partial\mathcal{M}, \qquad f_1 \overset{\Phi_1}{\mapsto} f_1.$$

Now, as Φ is an isotopy, we can produce the following map (which, as a consequence, is also an diffeomorphism):

$$\partial\mathcal{M} \times I \overset{\cong}{\to} \partial\mathcal{M} \times I, \qquad (x,t) \mapsto (\Phi_t(x), t)\,.$$

Here, $\partial\mathcal{M} \times I$ is a so-called "collar neighbourhood" of $\partial\mathcal{M}$. Recall that $\Phi_0 = \mathrm{id}$, and hence one can extend this diffeomorphism to all of $\mathcal{M}$ as the identity map in $\mathcal{M}\backslash(\partial\mathcal{M} \times I)$. Extend this to $\mathcal{M} \cup_{f_1} h_k$ as

$$\mathcal{M} \cup_{f_0} h_k \to \mathcal{M} \cup_{f_1} h_k, \qquad f_0(x) \mapsto \Phi\left(f_0(x)\right) = f_1(x).$$

□

Example 4.10. The n-sphere can be built by two n-disks: $S^n = D^n \cup_{\partial D^n} D^n$. Here, we can view one of the two disks as a 0-handle (which would be $D^0 \times D^n \cong D^n$), and the other as an n-handle (which would be $D^n \times D^0 \cong D^n$). We attach the n-handle to the 0-handle via $\partial D^n \times D^0 \cong \partial D^n \cong S^{n-1} \hookrightarrow \partial h_0 \cong \partial D^n \cong S^{n-1}$.

Example 4.11. One could consider a representation of the projective plane by the lower hemisphere, were we identify opposite points on the equator:

$$\mathbb{RP}^2 = \frac{\mathbb{R}^3\backslash\{0\}}{x \sim \lambda x} \cong \frac{S^2}{x \sim -x}.$$

Remark 4.12. Representing objects by handle decompositions can in general not be done in an unique way. For example, regarding , the following handle decomposition would also suffice:

Definition 4.13. Let $\mathcal{M}$ be a closed n-manifold. A handle decomposition for $\mathcal{M}$ is a sequence of n-manifolds

$$\varnothing \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_n = \mathcal{M}$$

such that $\mathcal{M}_{i+1}$ arises from $\mathcal{M}_i$ by attaching $(i+1)$-handles with disjoint attaching regions.

Lemma 4.14. *Any manifold admits a handle decomposition.*

Proof. This is a direct consequence of the fact that any manifold admits a triangulation[4]. □

[4]Result proved by J.H.C. Whitehead (1930's).

Also, it can be useful to know that any handle decomposition h_k admits a dual handle decomposition h_{n-k}, which are diffeomorphic to each other:

$$h_k := D^k \times D^{n-k} \cong D^{n-k} \times D^k =: h_{n-k}.$$

The attaching maps for the dual handle decomposition are determined by the gluing of $(k+1)$-handles. To determine the dual of a certain handle decomposition, you "have to turn it upside-down," as the famous Jorge Becerra once said.

Exercise. Determine the dual handle decompositions of the examples given above.

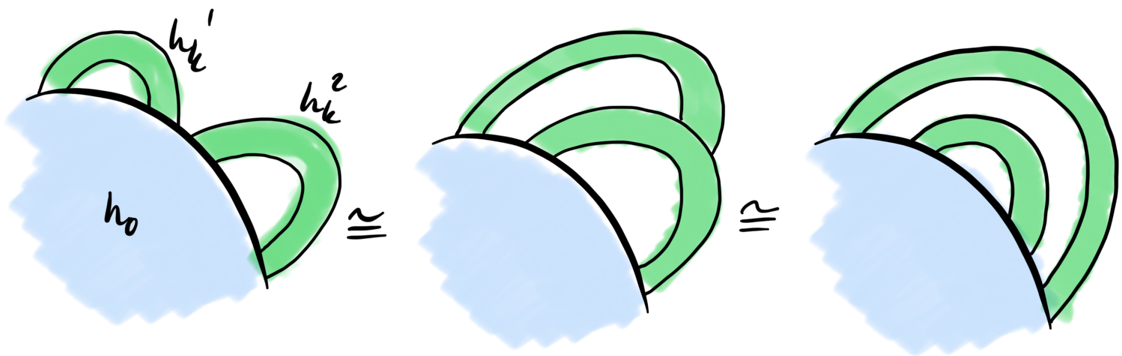
Lemma 4.15. *Any closed, connected n-manifold has a handle decomposition with a single 0-handle and a single n-handle.*

Proof. If we consider any (non-trivial; existing of more than one 0-handle) handle decomposition, we have that $\mathcal{M}_0 = \bigsqcup_n D^n$, and $\mathcal{M}_1 = \mathcal{M}_0 \cup (\bigcup h_1)$ – all 0-handles are connected by 1-handles, due to the fact we have a connected manifold. Topologically, such a structure is isomorphic to a single disk D^n. For the n-handles, one could consider the dual of the 0-handles and simply repeat the above argument. □

Now, a relevant question is: When are two handle decomposition related? When do they give rise to the same manifold?

Theorem 4.16 (Cerf's Theorem)**.** *Any two handle decompositions of a closed manifold are related by a finite sequence of the following moves:*

1. *Stabilisation/cancelling: If $0 < k \leqslant n$, an h_{k-1}- and h_k-handle can be cancelled (or created) provided that the attaching sphere of h_k and the belt sphere of h_{k-1} intersect transversally in a single point:*
$$\left(\partial D^k \times 0\right) \cap \left(0 \times \partial D^{n-(k-1)}\right) = *,$$
where the intersection must be transversal. "You should use h_k as a winch to topple h_{k-1}."
2. *Handle slide: Given two k-handles h_k^1, h_k^2 $(0 < k \leqslant n)$ isotope the attaching sphere of h_k^1 in $\partial\left(\mathcal{M} \cup h_k^2\right)$ by pushing it through a parallel of the core of h_k^2, which can be seen in the figure below.*



Exercise. Show that the argument used in the lemma about one 0-handle (and one n-handle) is an example of cancelling.

5 Classification of closed, connected, orientable surfaces I (Maurits Brinkman)

Here, merely as an introduction for the classification of closed, connected, orientable surfaces, we will look at some basic examples of handle decompositions. In particular, we will look at handle decompositions for S^1, S^2, T^2, handle decompositions for the connected sum of surfaces and the fact that any handle decomposition of a surface can be modified so that all 1-handles are attached to the 0-handle. All of these examples can be seen as useful tools to obtain a deeper understanding of the material treated in Lecture 2 (14/2): Handle decompositions above.

Example 5.1. The only closed, connected 1-manifold is S^1. Namely, considering Lemma 4.15 for $n = 1$, we see that any closed, connected 1-manifold $\mathcal{M}$ has a decomposition consisting of one 0-handle and one 1-handle. Hence, $\mathcal{M} = h_0 \sqcup h_1$, where $h_0 := D^0 \times D^1 \cong D^1$ and $h_1 := D^1 \times D_0 \cong D^1$. Both of these are intervals, which can be continuously deformed to half-circles, of which Figure 4 is the result.

J: The union is not disjoint, as the intervals intersect in their endpoints.

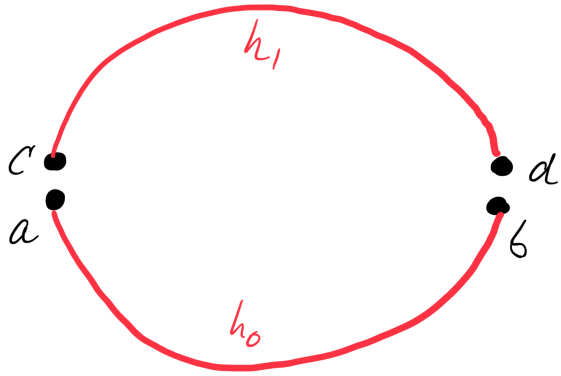


Figure 4: h_0 having boundary points a, b, and h_1 having boundary points c, d.

In general, as we know from Lecture 2 (14/2): Handle decompositions, one attaches a k-handle $h_k := D^k \times D^{n-k}$ to a manifold $\mathcal{N}$ using an embedding $\varphi : \partial D^k \times D^{n-k} \hookrightarrow \partial \mathcal{N}$. Remembering the construction sequence given in Definition 4.13, for $\mathcal{M}$ one needs to attach one 1-handle to the manifold that is the 0-handle, which is done by the map $\varphi : \partial D^1 \times D^0 \hookrightarrow \partial h_0$, where we know that $\partial D^1 \times D^0 \cong \partial D^1 = \{c, d\}$ for some two point a, b, and $\partial h_0 = \partial D^1 = \{a, b\}$. In short, we need to attach $\{a, b\}$ to $\{c, d\}$, which can be done in exactly two ways: we either attach the boundary points a to c and b to d, or we merge a with d and b with c. For both cases, we obtain S^1. □

Now, the following example is again an application of Lemma 4.15, where it is important to notice that we are allowed to take as many 1-handles as we like. The guarantee we have, is that there exists an decomposition that includes a single 0-handle and a single n-handle.

Example 5.2. Let's describe handle decompositions for S^2 and T^2 with one 0-handle and one 2-handle.

In order to construct S^2, we could attach a 2-handle $h_2 := D^2 \times D^0 \cong D^2$ to a 0-handle $h_0 := D^0 \times D^2 \cong D^2$ (both of these are disks, which can be continuously deformed into an upper and lower hemisphere, as depicted in Figure 5) via the embedding

$$\varphi : \partial D^2 \times D^0 \cong \partial D^2 = S^1 \hookrightarrow \partial h_0 \cong \partial D^2 = S^1.$$

The simplest case of this embedding is the identity map. This glues the two red parts depicted in Figure 5, from which then obtains S^2.

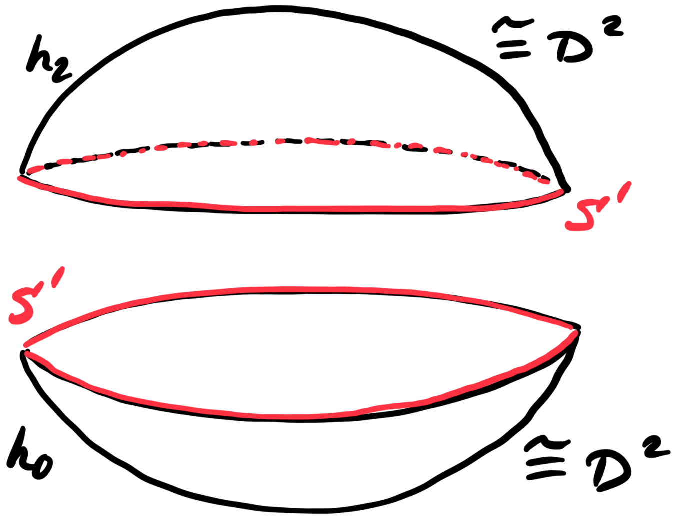


Figure 5: h_2 is attached to h_0, from which S^2 is formed.

In order to come up with a handle decomposition for T^2, consider its representation given on the left in Figure 6, where we identify opposite edges, and – as a consequence of that – the four vertices represent one and the same point. In terms of handle decompositions, these vertices and edges can be considered as two-dimensional objects: h_0's and h_1's, respectively.

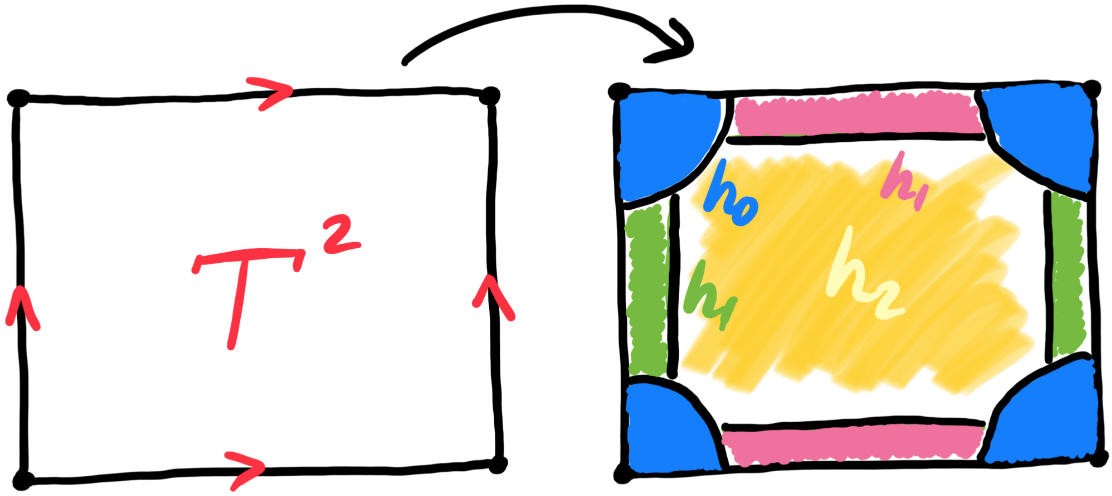


Figure 6: Deforming the torus into its handle decomposition.

Looking at the representation on the right side in Figure 6, one could merge the parts in blue together, as if this representation was a piece of paper, which we would fold into a ball:

The right representation in Figure 7, we recognise the attaching regions of the 1-handles (in pink and green) to the 0-handle (in blue), and the 2-handle (in yellow) which attaches to both the 0-handle and 1-handles. Finally, this leads to the sequence, as in Definition 4.13, depicted in Figure 8 below.

In particular, the former handle decomposition is created by attaching the 2-handle h_2 to $h_0 \sqcup h_1$ using the embedding $\partial D^2 \times D^0 \cong \partial D^2 = S^1 \hookrightarrow \partial (h_0 \sqcup h_1) \cong S^1$, for which we can again take the identity map. □

J: Rather than "vertices and edges can be considered as 2d objects", they can be thickened up to 2d objects

J: The RHS picture represents a neighbourhood of the 0-handle, write? Then the 1-handles should continue up-down and right-left; and the boundary of the yellow 2-handle should meet the 1-handles along their boundary, as in the LHS pic.

J: Again the union is not disjoint. Also can you write

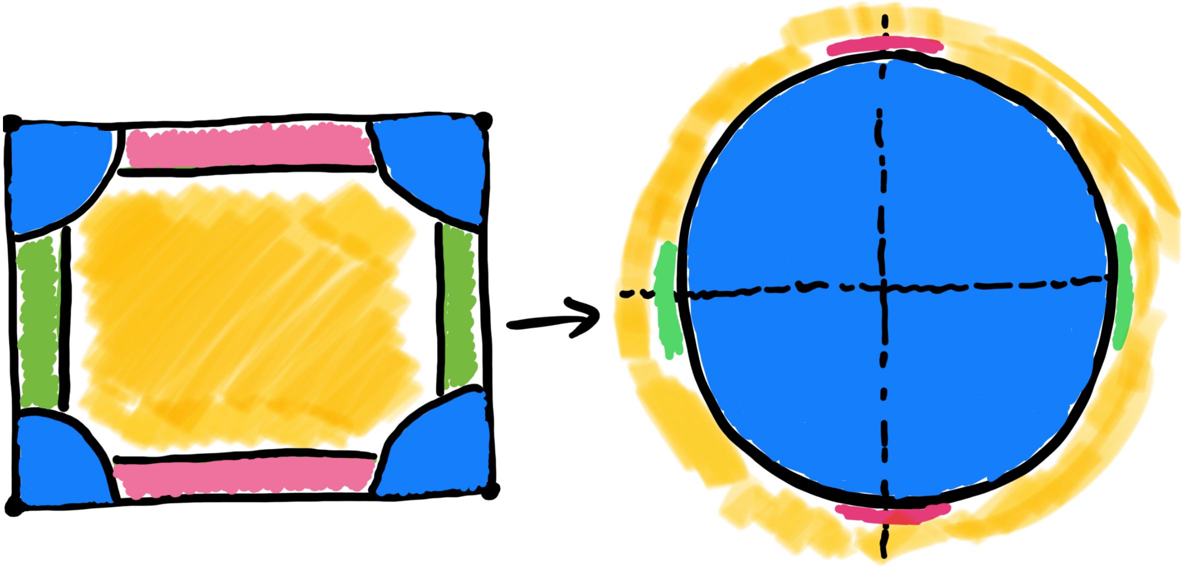


Figure 7: Deforming the torus representation.

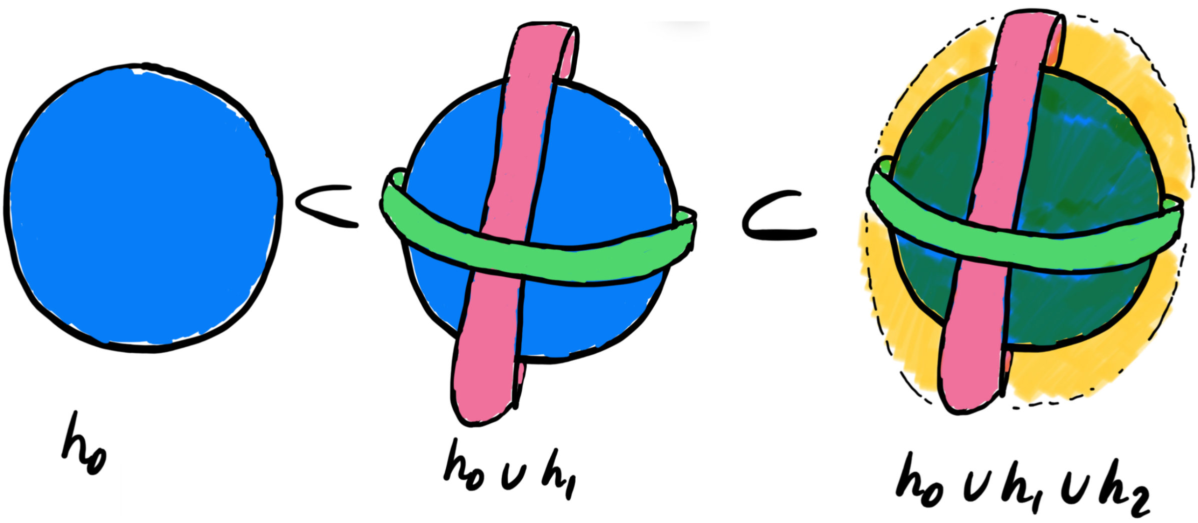


Figure 8: Creating a handle decomposition for T^2.

In the next example, we will look at a connected sum of surfaces, which is nothing more than drilling out tiny open disks of each of the surfaces one is considering, and then gluing these surfaces along the boundaries of the holes that are a result of the drilling, as depicted in Figure 9.

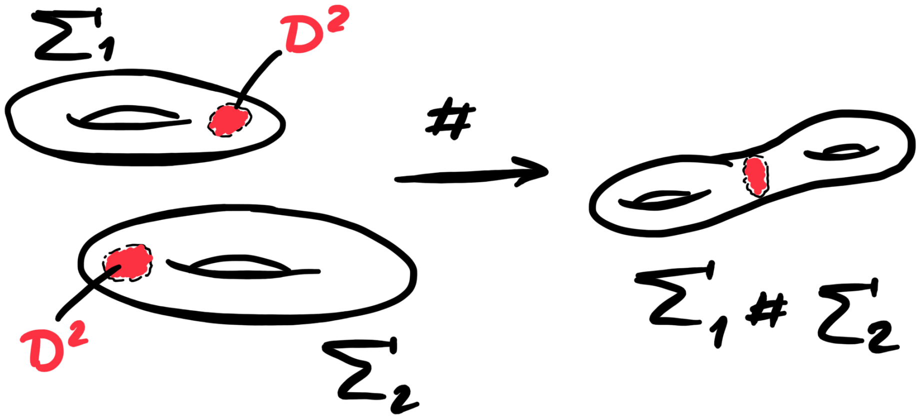


Figure 9: Example of a connected sum of two surfaces.

Example 5.3. In this example, we will describe a handle decomposition for the connected sum of surfaces, which we will assume to be closed (i.e., that the surfaces are compact and without boundary). For this, consider any two closed surfaces Σ_1 and Σ_2. The connected sum of Σ_1 and Σ_2 can be defined in the following – very informal – way:

$$\Sigma_1 \# \Sigma_2 = \frac{\left(\Sigma_1 - D^2\right) \sqcup \left(\Sigma_2 - D^2\right)}{\text{glue over } \partial\left(\Sigma_1 - D^2\right) \text{ and } \partial\left(\Sigma_2 - D^2\right)}.$$

Here, since we assumed Σ_1 and Σ_2 to have no boundary, it holds that $\partial\left(\Sigma_i - D^2\right) = \partial D^2 = S^1$ for $i = 1, 2$.

J: You have to remove the interiors of the discs.

Now, since we are considering surfaces, we are looking at 2-dimensional manifolds Σ_1, Σ_2, which – according to Lemma 4.15 – have a handle decomposition $h_0 \sqcup (\bigsqcup h_1) \sqcup h_2$, where $\bigsqcup h_1$ is the disjoint union of all 1-handles in this decomposition. Therefore, the handle decompositions of Σ_1 and Σ_2 will look as follows:

J: Again unions are not disjoint.

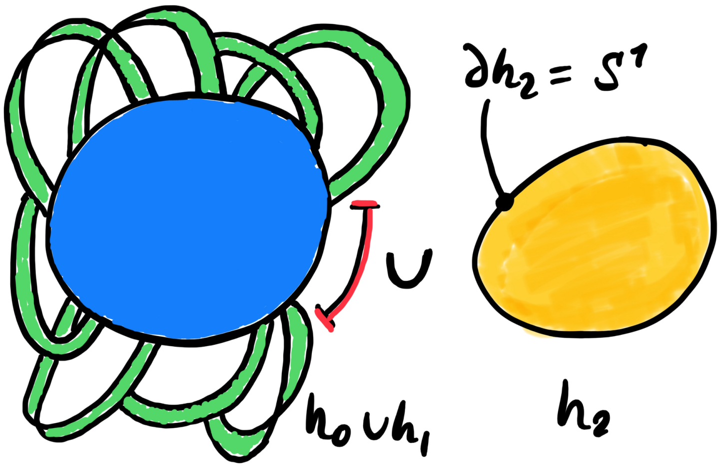


Figure 10: This is what the handle composition for both Σ_1 and Σ_2 looks like. The single 0-handle is coloured blue, the 1-handles in green and the single 2-handle in red.

Due to the fact that the 1-handles are attached to the 0-handle in a disjoint way, we can always find some ‘open space’, where no 1-handles are attached to the 0-handle. This open space can even be enlarged by continuous deformation; stretching the region indicated in red in Figure 10.

In order to obtain a connected sum, we would take a disk D^2 out of both Σ_1 and Σ_2. This is exactly what we will do for the handle decomposition, but now we are ‘cutting out’ the disks in a very clever way – we cut out the red regions depicted in Figure 11. These are indeed disks, as the h_2’s of both Σ_1 and Σ_2 are attached to the $h_0 \sqcup h_1$’s.

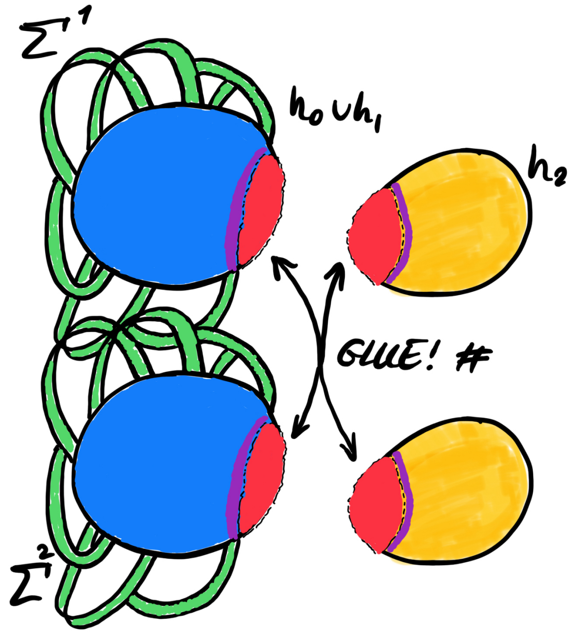


Figure 11: Glueing the handle decompositions of surfaces Σ_1 and Σ_2.

After this, we will glue along the parts in purple (in Figure 11), which are both copies of S^1, as explained above. This finally results in the following handle decomposition of $\Sigma_1 \# \Sigma_2$, where both 0-handles are depicted in blue, the 1-handles in green, and both 2-handles (which attach to both the 0-handles and 1-handles of the corresponding surfaces) in yellow:

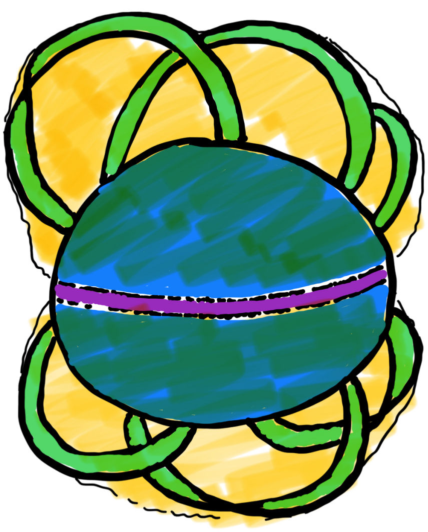


Figure 12: The handle decomposition of $\Sigma_1 \# \Sigma_2$.

Of course, we have only showed what the handle decomposition for a connected sum of two (closed) surfaces looks like, but one can repeat this procedure to obtain the handle decomposition for any amount of (closed) surfaces. □

The last theorem of this section looks deeper into Definition 4.13; it will look at a case that was previously not mentioned, merely to keep things simple. Namely, for a 1-handle it can happen that one of its 'legs' is attached to the 0-handle, while the other is attached to a 1-handle. By continuous deformation, we will show that this is not any different from both of the legs being attached to the 0-handle. In particular, the following theorem will be an application of Cerf's Theorem, which we discussed in Lecture 2 (14/2): Handle decompositions.

J: Rather than an application, it is the proof of the handle slide for $k = 1, n = 2$! :)

Theorem 5.4. *Any handle decomposition of a surface can be modified so that all 1-handles are attached to the 0-handle.*

Proof. Let us assume we are dealing with the following case, where we have a 1-handle that has one of its 'legs' attached to another 1-handle:

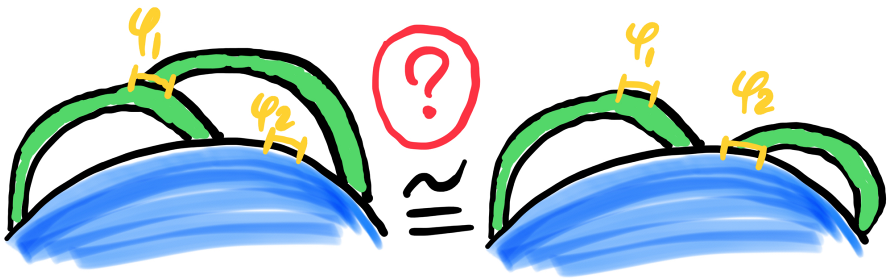


Figure 13: Visual representation of Theorem 5.4, where part of the 0-handle is shown (blue) and two 1-handles (green).

Now, for convenience, denote the region where we the leg of the 1-handle is attached to the other 1-handle by φ_1. It is our task to find a diffeomorphism that sends this region φ_1 to a region φ_2, where the latter is on the boundary of the 0-handle. Before we start building such a diffeomorphism, let us consider a part of the boundary (depicted as the pink $[a, b]$ in Figure 13) of our $\mathcal{M} := \mathcal{M}_1$ (see Definition 4.13) that contains both φ_1 and φ_2, and consider it as an interval of the real line, which can be done by the embedding $[a, b] \hookrightarrow \mathbb{R}$. In this copy of the real line, let us denote the distance that need to be covered to move φ_1 to φ_2 by $k \in \mathbb{R}$.

J: φ is the region or the embedding (attaching map)?

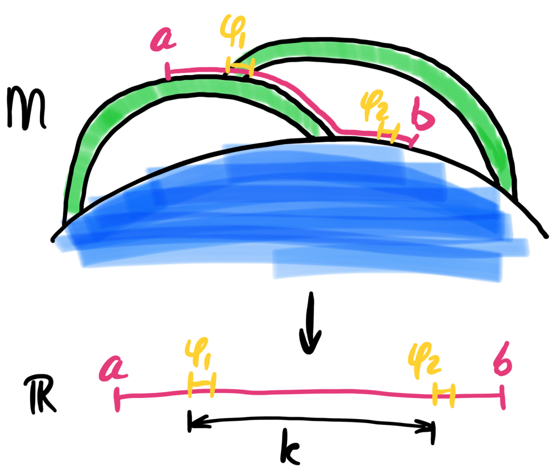


Figure 14: On our way to finding an isotopy H_t that moves φ_1 to φ_2.

In order to find the diffeomorphism that replaces the question mark in Figure 12, we will need to build the following isotopy (where $t \in I := [0, 1]$), which would move φ_1 to φ_2:

$$H_t : [a, b] \times I \to [a, b] \subset \partial\mathcal{M} \qquad \text{such that} \qquad H_0 = \varphi_1, \quad H_1 = \varphi_2.$$

For this isotopy, one could take $H_t = \varphi_1 + kt$. Now, as a consequence of the isotopy extension lemma, there exists an isotopy

$$\Phi_t : \partial\mathcal{M} \times I \xrightarrow{\cong} \partial\mathcal{M} \times I, \qquad (x, t) \mapsto (\Phi_t(x), t),$$

such that $\Phi_0(\varphi_1, 0) = 0$ and $\Phi_1(\varphi_1, 1) = \varphi_2$. We will take this diffeomorphism to be the identity map, except on the green area in Figure 14. □

J: Have a look whether you mean Φ or Φ_t. In general if $H : X \times I \to Y$ we put H_t for the map $H(-, t) : X \to Y$

6 Classification of closed, connected, orientable surfaces II (Kevin van Helden)

In this section, all functions are continuous and all surfaces are connected. From the previous section, we recall the following results:

Theorem 6.1. *Every surface has a handle decomposition with one 0-handle h_0, and with all 1-handles attached to ∂h_0.*

Proposition 6.2. *For every two strictly increasing functions $\phi_1, \phi_2 : [0,1] \to \mathbb{R}$, there exists a homeomorphism $h : \mathbb{R}^2_- \to \mathbb{R}^2_-$ (with $\mathbb{R}^2_- := \{(x,y) \mid y \leqslant 0\}$ and $\partial\mathbb{R}^2_- \cong \mathbb{R}$) such that $h \circ \phi_1 = \phi_2$ and such that h is equal to the identity outside of a compact set of $\mathbb{R}^2_-$.*

Proposition 6.3. *Let F be a surface. For $\varphi_1, \varphi_2 : \partial D^1 \times D^1 \to \partial F$ and a homeomorphism $h : F \to F$ such that $h \circ \varphi_1 = \varphi_2$, we have that $D^1 \times D^1 \cup_{\varphi_1} F = D^1 \times D^1 \cup_{\varphi_2} F$.*

Locally, we can view the boundary of a surface F as $\mathbb{R}^2_-$. Applying 6.2 to such a neighbourhood of F, we find that we can slide an attaching region of a 1-handle along the boundary of F without changing the surface (up to homeomorphism, that is.)

We will now proceed to define the object that this section is about: Kirby diagrams.

Definition 6.4. A **Kirby diagram** is a sequence $(a_i)_{i=1}^{2n}$ such that $|\{a_i = a_j \mid 1 \leqslant i \leqslant n\}| = 2$ for all $1 \leqslant j \leqslant n$.

We normally denote a Kirby diagram by a line and attach a_i to the point $f(a_i)$ for a strictly increasing function $f : \{1, \dots, n\} \to \mathbb{R}$.

Example 6.5. An example of a Kirby diagram is $K = (1,3,2,3,2,4,4,1)$, or as in Figure 15.

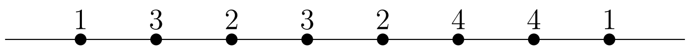


Figure 15: The Kirby diagram K.

Kirby diagrams are a way to encode an orientable surface. We recover an orientable surface F_K from a given Kirby diagram K as follows:

1. Use the 1-point compactification on $\mathbb{R}$ to turn it into S^1. The disk h_0 enveloped by S^1 is our 0-handle.
2. Attach 1-handles in an orientation preserving way to h_0, such that the boundary points of the cores of the 1-handles coincide on the Kirby diagram. We call this new surface h_1.
3. Attach a 2-handle to every boundary component of h_1. The new surface is F_K.

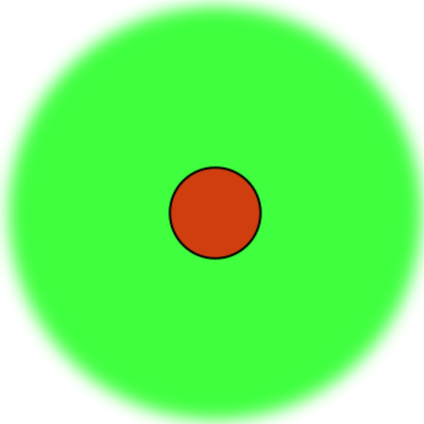
Example 6.6. The empty sequence leads to the surface in Figure 16a, which is S^2. The Kirby diagram $K = (1,2,1,2)$ leads to the surface in Figure 16b, which is the torus T^2.

It might not be evident that the algorithm displayed before yields a unique surface (up to homeomorphism). We will now prove that this is in fact the case.

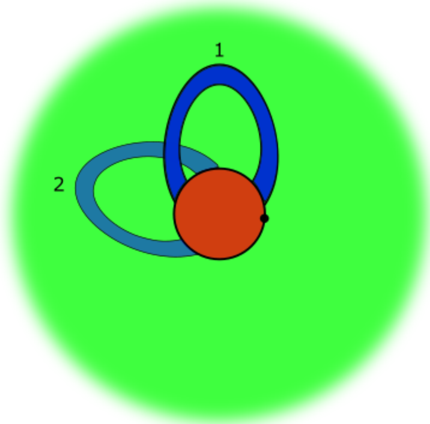
Lemma 6.7. *Let K be a Kirby diagram. Then F_K is well-defined (up to homeomorphism).*

Proof. The only ambiguity that can arise is in step 1, as it is not clear how many twists a 1-handle can get before we attach it to h_0. As we want F_K to be orientable, the number of twists in each handle should be even. By direct application of Proposition 6.3, we find that the two surfaces in Figure 17 are homeomorphic, and thus h_1 is well-defined (up to homeomorphism), whatever the number of orientation preserving twists. Glueing 2-handles along the boundary components of h_1 does not change the homeomorphism type of F_K either.

□

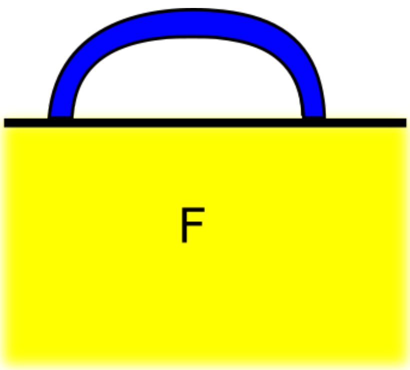


(a) The surface corresponding to the empty Kirby diagram.

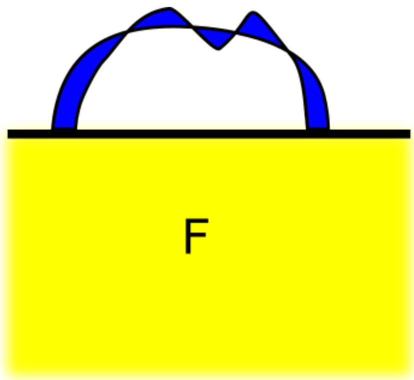


(b) The surface corresponding to $K = (1, 2, 1, 2)$.

Figure 16: In this figure, all 0-,1- and 2-handles are red, blue and green respectively.



(a) Attaching a 1-handle to a surface F without twists.



(b) Attaching a 1-handle to a surface F with two twists.

Figure 17: In this figure, the surface F is coloured yellow and the attaching 1-handle blue.

We could also reverse the process we described above: starting with a surface F, we can obtain a Kirby diagram as follows.

1. Consider a handlebody decomposition of F with only 0-handle h_0 and such that all 1-handles are attached to ∂h_0.

2. Pick a point x on $\partial h_0 \cong S^1$ that is not in the attaching region of any 1-handle.

3. Delete the point x from S^1 and denote every attaching region by the 1-handle that is attached to it. This results in a Kirby diagram.

Remark 6.8. If we want to consider the connected sum of two surfaces F_1, F_2, we can do this by eliminating a neighbourhood of the point x_1 and x_2 that does not intersect with the attaching region of any 1-handle. Then glueing the newly created boundaries of the 0-handles and of the 2-handles together, we obtain a surface which we can give a Kirby diagram determined by concatenating the Kirby diagrams of F_1 and F_2 obtained by deleting x_1 and x_2 respectively.

Remark 6.9 (Warning!)**.** The above process to create a Kirby diagram from a surface is not unique! The two different handlebody decompositions of S^2 in Figure 18 yield different Kirby diagrams, from left to right $(1, 1, 2, 2)$ and $(1, 2, 2, 1)$.

(a) A handlebody decomposition of S^2. (b) A handlebody decomposition of S^2.

Figure 18: Two different handlebody decomposition of S^2. In this figure, all 0-,1- and 2-handles are red, blue and green respectively.

Definition 6.10. We define the relation $\sim$ between two Kirby diagrams A and B by stating that $A \sim B$ if and only if F_A and F_B are homeomorphic.

It is not hard to see that the above relation is an equivalence relation. We will henceforth denote the equivalence class of A by $\overline{A}$.

The following three 'moves' do not change to homeomorphism type of a surface. Note that we will not write additional 1-handles, but the Kirby diagram should make clear where possible extra one-handles could be attached.

1. The creation/annihilation of handle pairs, as described in Figure 19. This implies that $\overline{(\dots, a, a, \dots)} = \overline{(\dots)}$.

2. Crossing through the deleted point, as described in Figure 20. This implies that $\overline{(a, \dots)} = \overline{(\dots, a)}$.

(a) Annihilation of a handle pair. (b) Creation of a handle pair.

Figure 19: Creation/annihilation of handle pairs. In this figure, all 0-,1- and 2-handles are red, blue and green respectively.

(a) Crossing through the deleted point. (b) Crossing through the deleted point.

Figure 20: Creation/annihilation of handle pairs. In this figure, all 0-,1- and 2-handles are red, blue and green respectively.

(a) Annihilation of a handle pair. (b) Creation of a handle pair.

Figure 21: Creation/annihilation of handle pairs. In this figure, all 0-,1- and 2-handles are red, blue and green respectively.

3. A handle slide, as described in Figure 21. The two spaces described in Figure 21a and 21b are homeomorphic by Proposition 6.2 and 6.3, as we move 1-handle b along the boundary of h_0 and a. This implies that $\overline{(b,\dots,b,a\dots,a)} = \overline{(b,\dots,a\dots,a,b)}$.

Ruben IJ: Shouldn't there be dots at both ends of these sequences,i.e. $(\dots, b, \dots, b, a, \dots$ and so on?

We are now ready to classify the closed orientable surfaces.

Theorem 6.11. *Every closed orientable surfaces is isomorphic to* $\#_k T^2$ *for some* $k \in \mathbb{N}$ *(with* $\#_0 T^2 := S^2$*).*

Proof. Let $A = (a_1, \dots, a_{2n})$ be a Kirby diagram of a surface F. We will prove the theorem by induction on the number n of 1-handles attached to h_0.
If $n = 0$ or $n = 1$, then A is empty or $A = (1, 1)$ and by creation/annihilation of a handle pair and by Example 6.6, we then know that $F \cong S^2 = \#_0 T^2$.
Now assume that $n > 1$. Then pick a $a = a_i = a_j$ such that $a_k \neq a_l$ for $i < k < l < j$ (if we cannot, we can annihilate a 1-handle and the induction step is complete). Then (by possibly crossing through the deleted point) there is a b such that we write $\overline{A} = \overline{(\dots,a,\dots,b,\dots,a,\dots,b,\dots)}$. By handle slides along the inner side of the 1-handle a, we can move all the elements between the left-most b and the right-most a to the (direct) right of the left-most a, meaning that we get that

$$\overline{A} = \overline{(\dots,a,\dots,b,\dots,a,\dots,b,\dots)} = \overline{(\dots,a,\dots,b,a,\dots,b,\dots)}. \tag{1}$$

By handle sliding the elements between the left-most a and b along b and by handle sliding the elements between the right-most a and b along a, we get that

$$\overline{A} = \overline{(\dots,a,\dots,b,a,\dots,b,\dots)} = \overline{\dots,a,b,a,b,\dots}. \tag{2}$$

This gives us that F is homeomorphic to the connected sum of T^2 and a surface with $(n-2)$ 1-handles. This also completes the induction step. □

Exercises

1. Prove that $\#_k T^2$ is homeomorphic to $\#_l T^2$ if and only if $k = l$.
2. Prove the Poincaré conjecture in dimension two: any closed orientable surface homotopy equivalent to S^2 is (homeomorphic to) S^2.
3. We described an algorithm to obtain an orientable surface of a Kirby diagram. Is there also a way to obtain non-orientable surfaces? If so, can you classify the closed non-orientable two-surfaces?

7 Lecture 3 (21/2): 3-manifolds

Given the rather simple classification of closed oriented surfaces as connected sums of finitely many tori one may hope that something similar happens in dimension three. One way to explore this is to consider what kind of manifolds are obtained by gluing together finitely many tetrahedra along their faces. The freely available program Regina does precisely that. It contains a complete list of all closed 3-manifolds that can be obtained by gluing the faces of at most ten tetrahedra and the results are baffling. There are manifold manifolds and the reader is urged to play around with the program to see what is going on.

Some general observations are that at first one sees many lens spaces $L(p,q)$ see a later lecture. After the lens spaces we also see so called Seifert fibered spaces, which include products like $\Sigma \times S^1$ where Σ is a closed orientable surface. More generally Seifert fibered spaces are always a union of circles but there are finitely many exceptional circles where the other circles wind around the circle as happens in the lens space.

As the number of tetrahedra increases new phenomena start to occur and we see more and more hyperbolic 3-manifolds. This hints at the fact that the classification of 3-manifolds is far from simple. One way to come to terms with 3-manifolds is to use Heegaard splittings and surgery to express 3-manifolds in terms of lower-dimensional objects such as surfaces and knots.

Keeping in mind the Cerf theorem that any closed orientable manifold has a handle decomposition we start by considering only the 0 and 1 handles.

Definition 7.1. A 3-dimensional handlebody of genus g is the attachment of g one-handles to a single 0-handle.

Interpreting the remaining 2-handles and 3-handle as making up a handlebody of the same genus we arrive at a Heegaard splitting:

Definition 7.2. A Heegaard splitting is a 3-manifold of the form $H \cup_h H'$ where H, H' are handlebodies of the same genus and $h : \partial H \to \partial H'$ is a diffeomorphism.

Handlebodies in dimension three are pretty simple things but the diffeomorphism h we are gluing along can be highly complicated. One way to generate interesting diffeomorphisms from a surface to itself is using Dehn twists.

Definition 7.3. For any embedding α of $A = S^1 \times [0,1]$ into closed surface Σ define the Dehn twist $\tau_\alpha : \Sigma \to \Sigma$ to be the following diffeomorphism. Outside of the image of α we have $\tau_\alpha = id$. Parametrizing A by $(e^{2\pi i\theta}, t)$ we set $\tau_\alpha(\alpha(e^{2\pi i\theta}, t)) = \alpha(e^{2\pi i(\theta+t)}, t)$.

The exact details of how big the annulus A defining the Dehn twist is does not interest us much. We know that the 3-manifold obtained from a Heegaard splitting does not change (up to diffeomorphism) when we change the gluing map h by an isotopy. This motivates the study of the diffeomorphisms up to isotopy, known as the mapping class group.

Definition 7.4. For any orientable closed surface Σ denote by $MCG(\Sigma)$ the set of isotopy classes of orientation preserving diffeomorphisms from Σ to itself.

Surprisingly the Dehn twists generate the mapping class group. This is not proved easily and there are many interesting relations between the Dehn twists but we will not get into this in this course.

Theorem 7.5. *The mapping class group of a surface of genus g is generated by Dehn twists long finitely many curves.*

Heegaard splittings are a useful way to represent 3-manifolds but sometimes it is pleasant to pass from gluing handlebodies to gluing solid tori only. This is known as surgery and we will have much to say about it in the future.

Definition 7.6. For a 3-manifold M, an embedded closed solid torus $K \subset M$ and a diffeomorphism $h : \partial(S^1 \times D^2) \to \partial M - int(K)$ define the surgery $M(K, h) = S^1 \times D^2 \cup_h M - int(K)$.

Surgery is closely related to Dehn twists.

Theorem 7.7. *Suppose we have a Heegaard splitting $M = H \cup_h H'$ of genus g and a simple closed curve α on ∂H. Then $M(K, \phi) = \cup_{\tau_\alpha \circ h} H'$, where K is the thickening of the curve α and ϕ sends the meridian of to some curve on ∂K.*

Through this theorem the existence of handle decompositions implies any 3-manifold has not only a Heegaard splitting but also a description as the repeated surgery of S^3 along several knots.

8 The lens spaces $L(p,q)$ (Ruben van Dijk)

Lens spaces are generally considered the simplest (nontrivial) type of 3-manifolds, allowing for explicit descriptions and some degree of visualisation that Dehn surgery generally does not provide. Furthermore, lens spaces offer examples of manifolds that have the same homology and are homotopy-equivalent, but are not homeomorphic.

Definition 8.1 (Lens spaces)**.** Let $p, q \in \mathbb{Z}_{>0}$ be coprime integers, and define the following diffeomorphism on the sphere $S^3 = \{(z_1, z_2) \,:\, |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$:

$$\begin{aligned} \tau_{p,q} \colon S^3 &\longrightarrow S^3 \\ (z_1, z_2) &\longmapsto \left(e^{2\pi i/p} z_1,\ e^{2q\pi i/p} z_2\right). \end{aligned}$$

Given a choice of p and q, we say that $z, z' \in S^3$ are equivalent (denoted $z \sim z'$) if and only if there exists an $n \in \mathbb{Z}_{\geqslant 0}$ such that $\tau_{p,q}^n(z) = z'$, where $\tau_{p,q}^n = \tau_{p,q} \circ \tau_{p,q} \circ \cdots \circ \tau_{p,q}$ denotes n-times composition. The **lens space** of type (p,q) is then defined as the quotient space $L(p,q) := S^3/\sim$.

Remark 8.2. There is no conventional name for the map that we call $\tau_{p,q}$. Some authors omit the subscript and some do not give the map a name at all.

Remark 8.3. Algebraically, $\tau_{p,q}$ generates a group action of $\mathbb{Z}/p\mathbb{Z}$ on S^3, defined by $\overline{n}z = \tau_{p,q}^n(z)$. The lens space $L(p,q)$ consists of the orbits under $\mathbb{Z}/p\mathbb{Z}$.

This action is free, meaning that the stabiliser subgroup $\{g \in \mathbb{Z}/p\mathbb{Z} : gz = z\}$ is trivial; this is obvious from the fact that $\overline{n}z = \tau_{p,q}^n(z) = z$ only if n is a multiple of p.

The freeness of the action also follows from the fact that it is properly discontinuous, i.e. every $z \in S^3$ has a neighbourhood U_z disjoint from its image gU_z for all nontrivial $g \in \mathbb{Z}/p\mathbb{Z}\backslash\{\overline{1}\}$. Indeed, if $p \neq 2$, let

$$U_z := \left\{(w_1, w_2) \in S^3 : |w_1 - z_1|^2 + |w_2 - z_2|^2 < \tfrac{1}{4}\sin^2(2\pi/p)\right\}$$

be the intersection of S^3 with an open 4-disk of radius $\sin(2\pi/p)/2$ around $z = (z_1, z_2)$. Then its image under $\tau_{p,q}^n$ is S^3 intesected with a ball of the same radius around $\tau_{p,q}^n(z)$:

$$\begin{aligned} \tau_{p,q}^n(U_z) &= \left\{\left(e^{2n\pi i/p} w_1, e^{2nq\pi i/p} w_2\right) \in S^3 : |w_1 - z_1|^2 + |w_2 - z_2|^2 < \tfrac{1}{4}\sin^2(2\pi/p)\right\} \\ &= \left\{(w_1, w_2) \in S^3 : \left|w_1 - e^{2n\pi i/p} z_1\right|^2 + \left|w_2 - e^{2nq\pi i/p} z_2\right|^2 < \tfrac{1}{4}\sin^2(2\pi/p)\right\}. \end{aligned}$$

Since the distance between z and its image is, for $n \not\equiv 0 \bmod p$,

$$\left|z - \tau_{p,q}^n(z)\right| = \sqrt{\left|z_1 - e^{2n\pi i/p} z_1\right|^2 + \left|z_2 - e^{2nq\pi i/p} z_2\right|^2} \geqslant \sin(2\pi/p),$$

the open disks of radius $\sin(2\pi/p)/2$ around z and $\tau_{p,q}^n(z)$ do not intersect. If $p = 2$, simply pick a ball of radius 1.

The quotient space of a manifold under a free and properly discontinuous group action of diffeomorphisms is again a smooth manifold, yielding Proposition 8.4. A detailed discussion about quotient spaces constructed via group actions can be found in e.g. Lee's *Introduction to Smooth Manifolds* and Van der Ban's *Notes on quotients and group actions.*

Proposition 8.4. *The lens space $L(p,q)$ is a closed, connected, orientable smooth 3-manifold for any $p, q \in \mathbb{Z}_{>0}$.*

J: A quick argument for orientability is the following: the projection map $S^3 \to L(p,q)$ is a local diffeomorphism, since the action is properly discontinuous. Hence pointwise the differential is an isomor-

Example 8.5. For any positive integer q, $\tau_{1,q}$ is the identity on S^3, so $L(1,q) \cong S^3$. Some authors exclude S^3 from the definition of a lens space by requiring that $p > q$.

Example 8.6. Consider $\tau_{2,1}$, which maps points on S^3 to their antipodes. Hence, $L(2,1)$ identifies opposite points on S^3; using the usual correspondence $\mathbb{C}^2 \cong \mathbb{R}^4$, this implies that $L(2,1) \cong \mathbb{RP}^3$.

Theorem 8.7. *For any coprime $p, q \in \mathbb{Z}_{>0}$, the lens space $L(p,q)$ is diffeomorphic to the $-p/q$-surgery on the unknot in S^3.*

Proof. Via a stereographic projection, we have a diffeomorphism $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$, in which the circles $\{(z_1,z_2) : |z_1| = 1\}, \{(z_1,z_2) : |z_2| = 1\} \subset S^3$ again correspond to circles, as in Figure 22; notice that the image of $\{|z_1| = 1\}$ goes through the point at infinity.

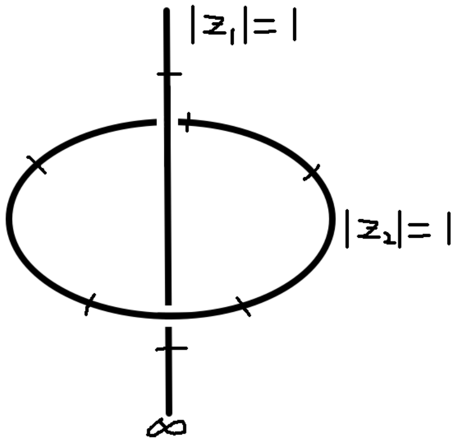


Figure 22: The images of $\{|z_1| = 1\}, \{|z_2| = 1\} \subset S^3$ under stereographic projection.

We divide each circle into p equally sized arcs (on $\{|z_1| = 1\}$ these are projected to line segments). In $L(p,q)$, the points on one of these arcs are identified with points on any other. Hence, we can discard all but one segment of $\{|z_1| = 1\}$ without losing any points of $L(p,q)$. By the same reasoning, all points of $L(p,q)$ are contained in a ball whose poles are the end points of a segment of $\{|z_1| = 1\}$ and whose equator is $\{|z_2| = 1\}$.

No two points in the interior of this ball are equivalent, but on the spherical boundary, each point in the upper hemisphere is identified with its image after a $2\pi q/p$-rotation around $\{|z_1| = 1\}$, followed by reflection into the lower hemisphere; see Figure 23.

J: Could you explain why this claim is true?

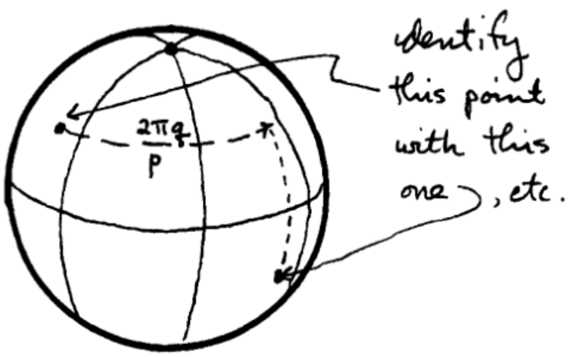


Figure 23: The identification on S^2 that turns D^3 into $L(p,q)$.

Now, we core this ball, removing a solid cylinder centered around $\{|z_1| = 1\}$. We call this solid cylinder V_1, and the closure of what remains is denoted by V_2. See Figure 24 on the next page.

We first look at V_2, which we split into p parts as shown in Figure 25. Using the identification we had on the boundary of the original sphere, we glue them back into a single solid, whose top

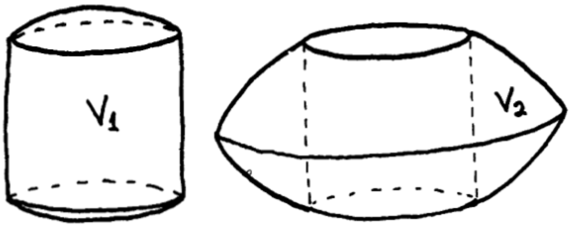


Figure 24: The splitting of $L(p, q)$ into V_1 and V_2.

and bottom are identified – this is a torus. A meridian on this torus runs along the side of the previous solid, which corresponds to p equidistant vertical lines along the boundary of V_1.

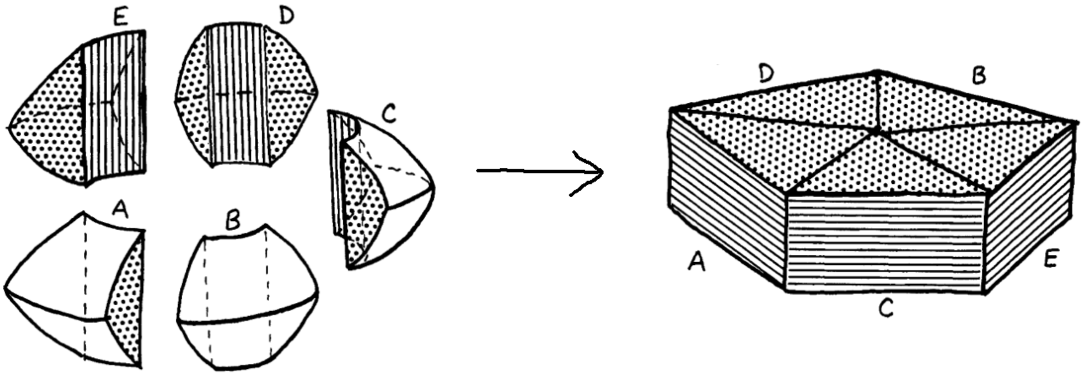


Figure 25: Disassembling and reassembling V_2, here shown for $p = 5$ and $q = 2$.

Now, identifying the top and bottom of V_1, we obtain another torus. The identification gives it a $2\pi q/p$ twist, causing the meridian lines to connect into a single curve that runs along the meridian of the V_1-torus q times and along its longitude p times. Gluing V_1 back into V_2 is therefore a $-p/q$-surgery, where the minus sign comes from the standard orientation of the longitude. □

Exercise 8.8. Show that the homology groups of $L(p, q)$ are given as follows:

$$H_k(L(p,q)) = \begin{cases} \mathbb{Z} & \text{if } k \in \{0,3\}, \\ \mathbb{Z}/p\mathbb{Z} & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the homology of $L(p, q)$ is independent of q.

Exercise 8.9. Show that $L(p, q) \cong L(p, q')$ if and only if $qq' \equiv \pm 1 \bmod p$ or $q \equiv \pm q' \bmod p$.

J: Probably computing $\pi_1(L(p,q) \cong \mathbb{Z}/p$ using van Kampen is easier.

Exercise 8.10. Show that $L(p, q)$ and $L(p, q')$ are homotopy-equivalent if and only if $\pm qq'$ is a quadratic residue modulo p, i.e. $\pm qq' \equiv x^2 \bmod p$ for some $x \in \mathbb{Z}$. Find p, q and q' such that $L(p, q)$ and $L(p, q')$ are homotopy-equivalent (and have the same homology), but are not homeomorphic.

J: Let op! These last two exercises are not exercises, they are rather difficult theorems (!)

9 Handle decomposition of $\mathbb{RP}^n$ and $\mathbb{CP}^n$ (Lisanne Sibma)

To discuss the handle decompositions of projective space, we first need to define what projective space is.

We define projective space as:

$$\mathbb{P}^n = (k^{n+1} \backslash \{0\}) / \sim,$$

where the equivalence relation is defined as

Definition 9.1.

$$[x_1, \ldots, x_n] \sim [y_1, \ldots, y_n] \iff \exists \lambda \in \mathbb{N} \text{ such that } x_1 = \lambda y_1, \ldots, \ \& \ x_n = \lambda y_n.$$

We denote the equivalence class of the point $[x, y, z]$ by $[x : y : z] \in \mathbb{P}^2$. We call these coordinates x, y, z the homogeneous coordinates of the point $[x, y, z]$.

We will also be using the idea of defining projective space in dimension n as the n-sphere, where we identify antipodal points, i.e. $x \sim -x$.

Handle decomposition of $\mathbb{CP}^n$ and $\mathbb{RP}^n$

The manifolds $\mathbb{RP}^n$ and $\mathbb{CP}^n$ each have a handle decomposition consisting of $n + 1$ handles. In the case of $\mathbb{RP}^n$, there is one handle of each index from 0 through n and in the case of $\mathbb{CP}^n$, there is one handle of each index from 0 through $2n$.

To construct such a handle decomposition for $\mathbb{RP}^n$, recall that $\mathbb{RP}^n$ is covered by $n + 1$ local parametrizations

$$\psi_i : \mathbb{R}^n \to \mathbb{RP}^n, i = 0, \ldots, n,$$

where ψ_i is given by

$$\psi_i(x_1, \ldots, x_n) = [x_1 : \cdots : x_i : 1 : x_{i+1} : \ldots x_n].$$

Although the images $\psi(\mathbb{R}^n)$ of the map above cover $\mathbb{RP}^n$, we claim that we can take a smaller domain and still cover all of $\mathbb{RP}^n$. Let this domain be $D^1 = [-1, 1]$, the unit interval. Define

$$B_i = \psi_i(D \times \cdots \times D),$$

taking n copies of D. Since we are working over the unit interval, we need to normalize each point $p \in \mathbb{RP}$ with homogeneous coordinates $[x_0 : \cdots : x_n]$ such that $\max_i |x_i| = 1$. Then

$$p \in B_i \iff |x_i| = 1 \qquad\qquad p \in \operatorname{int}(B_i) \iff |x_j| < 1 \ \forall j \neq i.$$

Now suppose we are given a point with $|x_i| < 1 \ \forall i$. Since we are working over an equivalence class of points, we can rescale all points in such a way that there is at least one coordinate equal to 1. Hence the B_i cover $\mathbb{RP}^n$.

Moreover, $p \in B_i \ \& \ p \in B_j \iff |x_i| = |x_j| = 1$, so the B_i only intersect along their boundaries.

We claim that B_k intersects $\cup_{i<k} B_i$ precisely on $\psi_k(\partial(D \times \cdots \times D) \times (D \times \cdots \times D))$ taking k copies of D in the first product (and hence $n - k$ in the second product). Hence we can interpret B_k as a k-handle attached to $\cup_{i<k} B_i$, with the attaching map being ψ_k.

When working over complex projective space $\mathbb{CP}^n$ we are using the same strategy, but instead of taking our domain to be D^1, we use the domain D^2, the unit disk.

The goal of the rest of this chapter is to describe the handle decompositions of $\mathbb{RP}^1, \mathbb{CP}^1, \mathbb{RP}^2$ and $\mathbb{RP}^3$. Moreover, we want to describe each attaching map as a composition of the maps ψ defined above.

Handle decomposition of $\mathbb{RP}^1$

The handle decomposition of $\mathbb{RP}^1$ consists of a single 0-handle and a single 1-handle. They are given by

- $h_0 := D^0 \times D^1 \cong D^1$,
- $h_1 := D^1 \times D^0 \cong D^1$.

To determine the attaching map, taking a look at Figure 26 below, we see that h_0 is mapped to the red part of S^1 under ψ_0 and h_1 is mapped to the green part of S^1 under ψ_1, where both $x, y \in [-1, 1]$. B_1 intersects B_0 precisely when their images coincide.

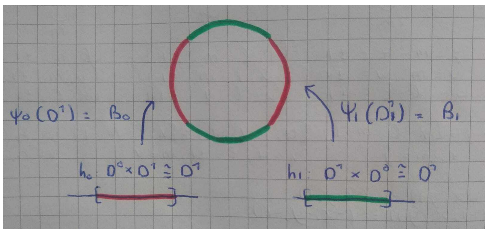


Figure 26: Commutative diagram for $\mathbb{RP}^1$

In other words, $B_1 \cap B_0$ if $\psi_0(D^1) = \psi_1(D^1)$. This gives us the equality

$$[1 : x] = [y : 1].$$

This equality holds if we take $y = \frac{1}{x} = x^{-1}$. As $|x| \leqslant 1$, we can only take $x = \pm 1$, because if $|x| < 1$ we have $|y| > 1$. We will use this knowledge without warning from now on. This gives us $y = \pm 1$, so B_1 intersects B_0 precisely in the points $[-1 : 1]$ and $[1 : 1]$.

So a map from h_1 to h_0 is well defined precisely when $x, y = \pm 1$, which gives us a map from the boundary of h_1 to the boundary of h_0. This is precisely our attaching map. Since the diagram in the picture above is a commutative diagram, we find that the attaching map is given by

$$\varphi = \psi_0^{-1} \circ \psi_1.$$

Let us look at where the attaching map sends the points $y = \pm 1$.

$$\varphi(1) = \psi_0^{-1}(\psi_1(1)) = \psi_0^{-1}([1 : 1]) = 1$$

$$\varphi(-1) = \psi_0^{-1}(\psi_1(-1)) = \psi_0^{-1}([-1 : 1]) = \psi_0^{-1}([1 : -1]) = -1$$

So our attaching map is precisely the identity map.

Handle decomposition of $\mathbb{CP}^1$

The handle decomposition of $\mathbb{CP}^1$ consists of a single 0-handle and a single 2-handle. They are given by

- $h_0 := D^0 \times D^2 \cong D^2$,
- $h_2 := D^2 \times D^0 \cong D^2$.

In contrast to real projective space, we define the balls B_i using the unit disk D^2 instead of the unit interval D^1. However, this doesn't necessarily change the procedure, because any point in the unit disk in complex space can be represented by a single complex number z. So again as above we solve

$$[1 : z] = [z' : 1].$$

So this equality holds for $z' = \frac{1}{z}$. Since $|z| \leqslant 1$, we find that $|z| = 1$. So the attaching region is precisely the boundary of the disk S^1 and the attaching map is given by $\psi_0^{-1} \circ \psi_1$. Let $z \in \mathbb{C}$ be such that $|z| = 1$. Then

$$\psi_0^{-1}(\psi_1(z)) = \psi_0^{-1}([z : 1]) = \psi_0^{-1}([1 : \frac{1}{zw}]) = \frac{1}{z}.$$

As we know, $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \bar{z}$ as $|z| = 1$. So we send each point on the unit circle to its complex conjugate. In other words, we attach the 2-handle by first flipping it around the real axis and then gluing it using the identity map.

Handle decomposition of $\mathbb{RP}^2$

The handle decomposition of $\mathbb{RP}^2$ consists of a single 0-handle, a single 1-handle and a single 2-handle. They are given by

- $h_0 := D^0 \times D^2 \cong D^2 \cong D^1 \times D^1$,
- $h_1 := D^1 \times D^1$,
- $h_2 := D^2 \times D^0 \cong D^2 \cong D^1 \times D^1$.

Similarly to the case of $\mathbb{RP}^1$, we can determine the attaching maps using a commutative diagram, which you can see in Figure 27 below

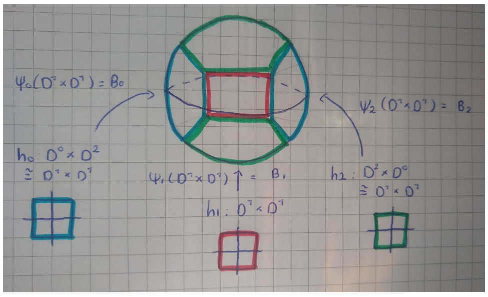


Figure 27: Commutative diagram of $\mathbb{RP}^2$

Attaching the 1-handle

We first need to attach the 1-handle to the 0-handle. We will be doing the same procedure as above, which means we are first looking at where B_1 intersects B_0, i.e. where $\psi_0(D \times D) = \psi_1(D \times D)$. This gives us

$$[1 : x : y] = [u : 1 : v].$$

This equality holds for $u = x^{-1}$ and $v = y \cdot x^{-1}$. We find that $u = \pm 1$ and $v = \pm y$. Since $y \in [-1, 1]$ already, we can say $v = y$, which we will also use from now on. So B_1 intersects B_0 in the points $[1 : 1 : y]$ and $[-1 : 1 : y]$. So a map from h_1 to h_0 is defined for pairs of points $(1, y)$ or $(-1, y)$ and is given by $\psi_0^{-1} \circ \psi_1$.

$$\psi_0^{-1}(\psi_1(1, y)) = \psi_0^{-1}([1 : 1 : y]) = (1, y).$$

$$\psi_0^{-1}(\psi_1(-1, y)) = \psi_0^{-1}([-1 : 1 : y]) = \psi_0^{-1}([1 : -1 : -y]) = (-1, -y)$$

So, the first piece is attached with the same orientation but the second piece is attached with a twist. Hence the result of attaching h_1 to h_0 is the Möbius band.

Attaching the 2-handle

Now, attaching the 2-handle we do in two parts. One part of the 2-handle attaches to the 0-handle and the other part attaches to the 1-handle.

To see where the 2-handle attaches to the 0-handle, we look at where B_2 intersects B_0, i.e. where $\psi_0(D \times D) = \psi_1(D \times D)$. This gives us

$$[1 : x : y] = [a : b : 1].$$

This equality holds for $a = y^{-1}$ and $b = xy^{-1}$. We find that $a = \pm 1$ and $b = \pm x = x$. So B_2 intersects B_0 in the points $[-1 : x : 1]$ and $[1 : x : 1]$. So a map from h_2 to h_0 is defined for pairs of points $(1, x)$ or $(-1, x)$ and is given by $\psi_0^{-1} \circ \psi_2$.

$$\psi_0^{-1}(\psi_2(1, x)) = \psi_0^{-1}([1 : x : 1]) = (x, 1).$$

$$\psi_0^{-1}(\psi_2(-1, x)) = \psi_0^{-1}([-1 : x : 1]) = \psi_0^{-1}([1 : -x : -1]) = (-x, -1).$$

To see where the 2-handle attaches to the 1-handle, we look at where B_2 intersects B_1, i.e. where $\psi_1(D \times D) = \psi_1(D \times D)$. This gives us

$$[u : 1 : v] = [a : b : 1].$$

This equality holds for $a = uv^{-1}$ and $b = v^{-1}$. We find that $a = \pm u = u$ and $b = \pm 1$. So B_2 intersects B_1 in the points $[u : 1 : 1]$ and $[u : -1 : 1]$. So a map from h_2 to h_1 is defined for pairs of points $(u, 1)$ or $(u, -1)$ and is given by $\psi_1^{-1} \circ \psi_2$.

$$\psi_1^{-1}(\psi_2(u, 1)) = \psi_1^{-1}([u : 1 : 1]) = (u, 1).$$

$$\psi_1^{-1}(\psi_2(u, -1)) = \psi_1^{-1}([u : -1 : 1]) = \psi_1^{-1}([-u : 1 : -1]) = (-u, -1).$$

To get an idea of where each part of the boundary of the 1-handle and 2-handle ends up, take a look at Figure 28 below.

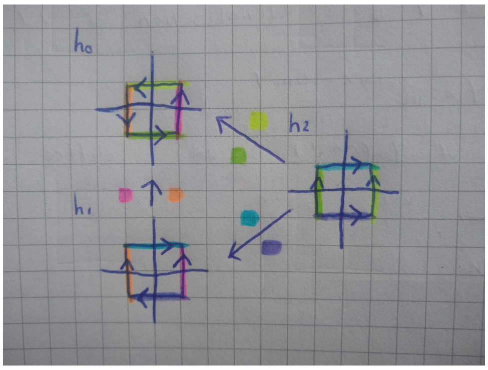


Figure 28: Attaching the 1-handle and the 2-handle

Handle decomposition of $\mathbb{RP}^3$

The handle decomposition of $\mathbb{RP}^3$ consists of a single 0-handle, 1-handle, 2-handle and 3-handle. They are given by

- $h_0 := D^0 \times D^3 \cong D^3 \cong D^1 \times D^1 \times D^1$,
- $h_1 := D^1 \times D^2 \cong D^1 \times D^1 \times D^1$,
- $h_2 := D^2 \times D^1 \cong D^1 \times D^1 \times D^1$,
- $h_3 := D^3 \times D^0 \cong D^3 \cong D^1 \times D^1 \times D^1$.

The idea for $\mathbb{RP}^3$ is the same as for $\mathbb{RP}^1$ and $\mathbb{RP}^2$, so we will now go through the process a bit faster.

Attaching the 1-handle

We first attach the 1-handle to the 0-handle. Solving $[1:x:y:z]=[u:1:v:w]$, we find that $u=\pm 1, v=y, w=z$. This gives us points $[\pm 1:1:y:z]$, so a map from h_1 to h_0 is defined for points $(\pm 1, y, z)$. The attaching map is given by the composition $\psi_0^{-1}\circ\psi_1$. Then

$$\psi_0^{-1}(\psi_1(1,y,z)) = \psi_0^{-1}([1:1:y:z]) = (1,y,z).$$

$$\psi_0^{-1}(\psi_1(-1,y,z)) = \psi_0^{-1}([1:-1:-y:-z]) = (-1,-y,-z).$$

Attaching the 2-handle

Attaching the 2-handle is done in two steps as before. To attach the 2-handle to the 0-handle, we solve $[1:x:y:z]=[a:b:1:c]$. This gives $a=\pm 1, b=x, c=z$. This gives us points $[\pm 1:x:1:z]$, so a map from h_2 to h_0 is defined for points $(\pm 1, x, z)$. The attaching map is given by $\psi_0^{-1}\circ\psi_2$. Then

$$\psi_0^{-1}(\psi_2(1,x,z)) = \psi_0^{-1}([1:x:1:z]) = (x,1,z).$$

$$\psi_0^{-1}(\psi_2(-1,x,z)) = \psi_0^{-1}([1:-x:-1:-z]) = (-x,-1,-z).$$

To attach the 2-handle to the 1-handle, we solve $[u:1:v:w]=[a:b:1:c]$. This gives $a=u, b=\pm 1, c=w$. This gives us points $[u:\pm:1:w]$, so a map from h_2 to h_1 is defined for points $(u,\pm 1,w)$. The attaching map is given by $\psi_1^{-1}\circ\psi_2$. Then

$$\psi_1^{-1}(\psi_2(u,1,w)) = \psi_1^{-1}([u:1:1:w]) = (u,1,w).$$
$$\psi_1^{-1}(\psi_2(u,-1,w)) = \psi_1^{-1}([-u:1:-1:-w]) = (-u,-1,-w).$$

Attaching the 3-handle

The procedure is probably clear now, so we will now only state what each attaching map does. First we attach the 3-handle to the 0-handle.

$$\psi_0^{-1}(\psi_3(1,x,y)) = \psi_0^{-1}([1:y:z:1]) = (y,z,1).$$

$$\psi_0^{-1}(\psi_3(-1,x,y)) = \psi_0^{-1}([1:-y:-z:-1]) = (-y,-z,-1).$$

Now we attach the 3-handle to the 1-handle.

$$\psi_1^{-1}(\psi_3(u,1,v)) = \psi_1^{-1}([u:1:v:1]) = (u,v,1).$$

$$\psi_1^{-1}(\psi_3(u,-1,v)) = \psi_1^{-1}([-u:1:-v:-1]) = (-u,-v,-1).$$

And last we attach the 3-handle to the 2-handle.

$$\psi_2^{-1}(\psi_3(a,b,1) = \psi_2^{-1}([a:b:1:1]) = (a,b,1).$$

$$\psi_2^{-1}(\psi_3(a,b,-1)) = \psi_2^{-1}([-a:-b:1:-1]) = (-a,-b,-1).$$

10 Lecture 4 (28/2): Orientability and Connected sum of surfaces

Orientability

Definition 10.1. Let M be an n-manifold. We say M is orientable if

$$\exists \omega \in \Omega^n(M) \text{ such that } \omega_x \neq 0 \ \forall x \in M.$$

An orientation is a choice of equivalence class $[\omega]$ of such ω, where

$$[\omega] = [\omega'] \iff \omega' = f\omega \text{ for } f > 0, f \in C^\infty(M).$$

An orientation in M is a continuous choice of positive basis in T_xM, $\forall x \in M$.

Remark 10.2. If M is connected, there are only two possible orientations. Given that M has an orientation, we denote by $-M$ the same manifold with the opposite orientation.

Example 10.3.
- n=1: An orientation is a choice of direction.
- n=2: An orientation is a choice of (counter)clockwise rotation.
- n=3: An orientation is a choice of right-hand rule/left-hand rule.

Proposition 10.4. *An n-manifold is non-orientable if and only if it contains an n-dimensional Möbius strip, i.e.*

$$\exists \text{Mob}_n \hookrightarrow M$$

with

$$\text{Mob}_n = D^{n-1} \times D^1/(x,0) \sim (r(x),1)$$

where $r : D^{n-1} \to D^{n-1}$ *is a reflexion*

Example 10.5. Let $n = 2$. Let us take the orientation to be a clockwise rotation. Then going around the Möbius strip, once we reach the beginning again, we find that we end up with a counterclockwise rotation. Hence if a 2-manifold with a Möbius band in it is not orientable.

Definition 10.6. Let $\varphi : M \to N$ be a local diffeomorphism (eg. an embedding) between oriented manifolds. We say that φ is

- orientation preserving if $\varphi_* = d\varphi$ sends a positive basis to a positive basis.
- orientation reversing if $\varphi_* = d\varphi$ sends a positive basis to a negative basis.

Connected sum

Definition 10.7. Let M_1, M_2 be two n-dimensional manifolds and let $\varphi_i : D^n \to M_i$ be embeddings. If both M_1, M_2 are oriented, assume WLOG that φ_1 preserves orientation and φ_2 reverses orientation. Then the connected sum of M_1 and M_2 along the embeddings φ_i is

$$M_1 \# M_2 = \frac{M_1 - \varphi_1(\mathring{D}^n) \sqcup M_2 - \varphi_2(\mathring{D}^n)}{\varphi_1(x) \sim \varphi_2(x)}$$

for $x \in \partial D^n$.

Remark 10.8. The choice of orientation is crucial. In general,

$$M_1 \# M_2 \not\cong M_1 \# (-M_2)$$

.

Example 10.9.

Proposition 10.10. *$M_1 \# M_2$ does not depend on the choice of embeddings.*

Theorem 10.11 (Disc theorem (Palais 60's))**.** *Let M be an n-dimensional connected manifold.*

1. *If M is non-orientable, then any two embeddings $D^n \hookrightarrow M$ are isotopic.*
2. *If M is orientable, but both embeddings either preserve or reverse orientation, they are also isotopic.*

Proof of Prop 10.10. Suppose M_1, M_2 are orientable. Let $\varphi_1, \overline{\varphi_1} : D^n \to M$ be two embeddings for M_1. Now suppose they both either preserve or reverse orientation, so they are isotopic by Theorem 10.11. Then, by the isotopy extension lemma 4.9 there is a map

$$\Psi : M_1 \times I \to M_1 \text{ s.t. } \overline{\varphi_1} = \Psi_1 \circ \varphi_1, \text{ where } \Psi_1 = \Psi(-,1).$$

Then

$$\Psi_1 : M_1 \xrightarrow{\cong} M_1.$$

This in turn implies

$$\Psi_1 : M_1 - \varphi_1(\mathring{D}^n) \xrightarrow{\cong} M_1 - \overline{\varphi_1}(\mathring{D}^n)$$

which completes the proof. □

Proposition 10.12. *For surfaces, the orientability requirement can be dropped at all times.*

Proof. For example, consider the connected sum of two tori, but with opposite orientation. Then the second torus is glued inside of the first one. However, we can 'pull' the second torus outside and we end up with what we would have gotten if the two tori had the same orientation. This is illustrated in Figure 29 below. In general, for surfaces, there is always an orientation reversing

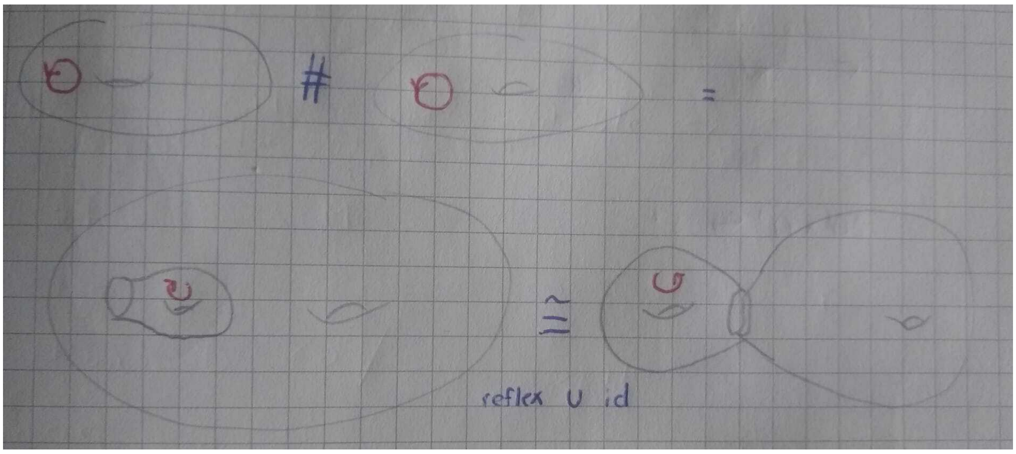


Figure 29: The gluing of two tori does not depend on orientation

diffeomorphism. □

Definition 10.13. Let M_1, M_2 be n-manifolds with one boundary component and let $\varphi_i : D^{n-1} \hookrightarrow \partial M_i$ be embeddings as before. Then the boundary sum of M_1 and M_2 is given by

$$M_1 \natural M_2 = \frac{M_1 \sqcup M_2}{\varphi_1(x) \sim \varphi_2(x)}$$

for $x \in D^{n-1}$.

Exercise 10.14. Show that $\partial(M_1 \natural M_2) = \partial M_1 \# \partial M_2$.

Determining embeddings

We first need to introduce a couple of concepts.

Remark 10.15. The orthogonal matrices are defined as

$$O(l) = \{l \times l \text{ real matrices A} \mid A^{-1} = A^T\} \subset GL(l) \subset \mathbb{R}^{l^2}$$

For example,

- $O(1) = S^0$
- $O(2) = S^1 \sqcup S^1$

In general

$$O(n) = SO(n) \sqcup SO(n)$$

.

Remark 10.16. For the definition of π_n, recall Definition (2.3). For now, we'll be using the following:

- $\pi_0(O(l)) \cong \mathbb{Z}/2\mathbb{Z}$.
- $\pi_1(O(l)) \cong \begin{cases} 0, & \text{if } l = 1. \\ \mathbb{Z}, & \text{if } l = 2. \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } l \geqslant 3. \end{cases}$

Remark 10.17. Let $\varphi : Y \hookrightarrow X$ be an embedding. Then $T_xY \subset T_xX, x \in Y$. The orthogonal complement of T_xY in T_xX is N_xY so that

$$T_xX = T_xY \bigoplus N_xY.$$

Then

$$NY = \bigsqcup_{x \in Y} N_xY$$

is the normal bundle of Y in X. A normal framing of Y is a trivialisation of NY, i.e.

$$NY \xrightarrow{\cong} Y \times \mathbb{R}^{dimX - dimY}.$$

Proposition 10.18. *Up to isotopy, an embedding $\varphi : \partial D^k \times D^{n-k} \hookrightarrow \partial M$ is completely determined (up to isotopy) by an embedding $\varphi_0 : \partial D^k \times 0 = S^{k-1} \hookrightarrow \partial M$ and a normal framing of $\varphi_0(S^{k-1})$ (both up to isotopy).*

The normal framing is is 1-1 correspondence with $\pi_{k-1}(O(n-k))$.

The normal framing of $\varphi_0(S^{k-1})$ is a diffeomorphism

$$N(\varphi_0(S^{k-1})) \xrightarrow{\cong} S^{k-1} \times \mathbb{R}^{n-k}.$$

Exercise 10.19. Show why we take $n-k$ in the power of $\mathbb{R}$ in the normal framing above.

Corollary 10.20. *To attach a 3-dimensional 2-handle to $H_g = h_0 \cup (\cup_g h_g)$, we only need to specify a curve.*

Proof. By Proposition 10.18, the embedding $\varphi : \partial D^2 \times D^1 =\hookrightarrow \partial H_g = \sum_g$ is determined by and embedding $\varphi_0 : \partial D^2 \times 0 = S^1 \hookrightarrow \sum_g$ and a normal framing, which is in 1-1 correspondence with $\pi_1(O(1))$. From Remark 10.16 we have that $\pi_1(O(1)) = 0$, so we indeed only have to specify the curve

$$\varphi_0 : \partial D^2 \times 0 = S^1 \hookrightarrow \sum_g$$

. □

11 Lecture 5 (7/3): Seifert manifolds

In this lecture, we elaborate on Lecture 3 by giving a more detailed description of what it means to perform *surgery along a framed link* in $\mathbb{R}^3$. Let us start by defining a *framed knot*. If $K : S^1 \hookrightarrow \mathbb{R}^3$ is a smooth embedding, then one can find an embedding $g : S^1 \times D^2 \hookrightarrow R^3$ such the core of the solid torus is mapped onto K. The embedded solid torus is called a *tubular neighbourhood* of K, denoted by νK. Let $T = \partial(\nu K)$ be the boundary torus. Up to isotopy, there exists a unique meridian m of T which is null-homotopic in νK, and there exist a unique longitude ℓ of T which is null-homotopic in $\mathbb{R}^3 \setminus \operatorname{int} \nu K$. Take m and ℓ to have the same common base point x_0. Together, m and ℓ generate the fundamental group of the torus $\pi_1(T, x_0) \cong \mathbb{Z} \times \mathbb{Z}$. The longitude ℓ is called the *canonical longitude*, and it is not necessarily a parallel copy of K on the torus T!

Now, any smooth embedding $\alpha : S^1 \to T$ determines, up to isotopy, a diffeomorphism $h : S^1 \times S^1 \to T$ that sends the meridian $\{0\} \times S^1$ to $\alpha(S^1)$. Identifying m and ℓ in $S^1 \times S^1$ under the embedding g of the tubular neighbourhood, there exist coprime integers p, q such that h is isotopic to the product of Dehn-twists $\tau_m^p \tau_\ell^q : S^1 \times S^1 \to T$.

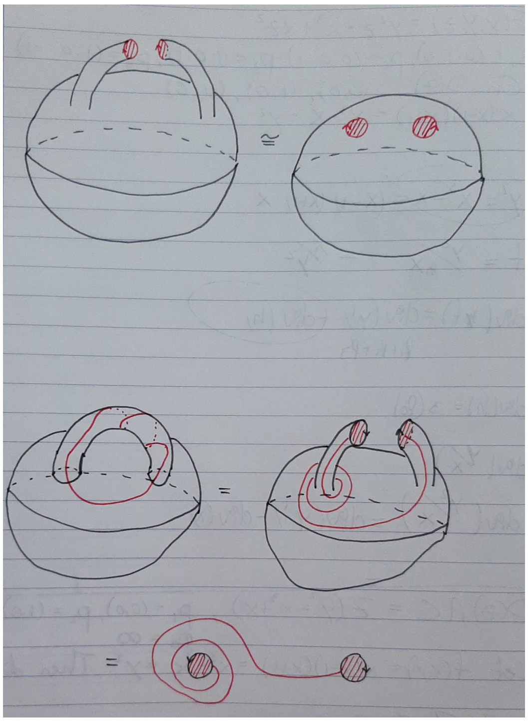


Figure 30: How to construct a Heegaard diagram.

12 Lecture 6 (10/3)

Recap of Heegaard diagrams Recall that we established an equivalence between the following two:

$$\{\text{Handle decompositions of 3-mfds}\} \iff \{\text{Heegaard Splittings}\}$$

This is done as follows. We can write $M = h_0 \cup (\cup h_1) \cup (\cup h_2) \cup h_3$. Then we take $H_g = h_0 \cup (\cup h_1)$, and after dualising this decomposition, we also get a genus g' handlebody $h_3^* \cup (\cup h_2^*) := H_{g'}$. They are attached to each other by a homeomorphism $f : \partial H_g \to H_{g'}$, whence $g = g'$ and $f \in \mathrm{MCG}(\Sigma_g)$. Thus, every handle decomposition determines a Heegaard splitting. Conversely, every Heegaard splitting obviously determines a handle decomposition. We also established the following equivalence:

$$\{\text{Embeddings } \partial D^1 \times D^1 \hookrightarrow \partial M\} \iff \{\text{Embeddings } \partial D^2 \hookrightarrow \partial M\} \times \pi_1(\mathrm{O}(1))$$

Since $\pi_1(\mathrm{O}(1)) = 0$, this term cancels.

Definition 12.1. A planar Heegaard diagram consists of $\mathbb{R}^2$, together with the attaching regions of 1-handles and attaching sphere of 2-handles, as illustrated in 30.

Example 12.2. See 31 for two equivalent diagrams which represent S^3.

We also reflect the fact that certain handle decompositions give diffeomorhic manifolds, by declaring that diagrams which are related by the following moves are equivalent:

1. Handle cancellation/creation
2. Handle slide (1-handle) 32
3. Handle slide (2-handle) 33

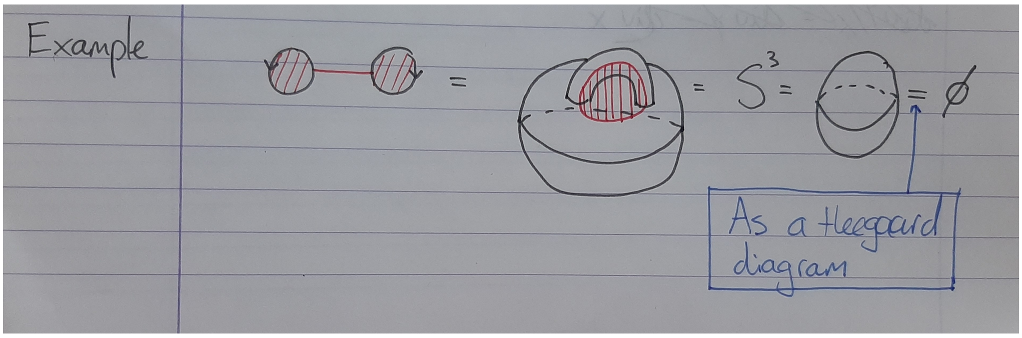


Figure 31: An example of a Heegaard diagram.

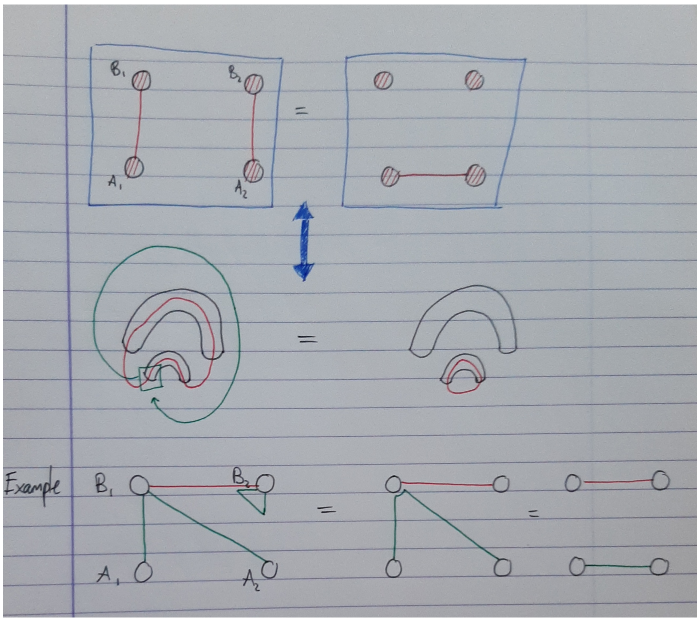


Figure 32: The handle slide move for a 1-handle

Finitely Presented Groups and 4-manifolds Finitely presented groups give rise to 4-manifolds.

$$G = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$$

Indeed, let such a group be given as above. To start with, we will construct a CW complex X which has $\pi_1(X) \cong G$, and then adapt this construction to 4-manifolds. Recall that a CW complex is inductively constructed by constructing the k-skeleta.

1. Set $X_0 = \{pt\}$.

2. For each generator g_i of G, attach a 1-cell, and obtain X_1 = Bouquet of n circles. It is easy to check using Van Kampen's theorem that $\pi_1(X_1) = F\langle g_1, \dots, g_n \rangle$, the free group on n generators.

3. Now, we want to impose the relations $r_1, \dots, r_m$. To do this, we will attach m 2-cells, as follows. Recall that a 2-cell is specified by giving a continuous map $\alpha_i : \partial D^2 = S^1 \to X_1$. Let α_i be the path represented by the relation r_i, viewed as a path in X_1. Then we obtain X_2 by attaching m 2-cells to X_1 with the specified attaching maps α_i, which impose the relations $\pi_1(X_2) \cong G$.

Now, we want to "thicken" this procedure to obtain an analogous result for 4-manifolds. We do this in the familiar way. We start with a 0-handle (i.e. $h_0 = D^4$) and attach 1-handles for

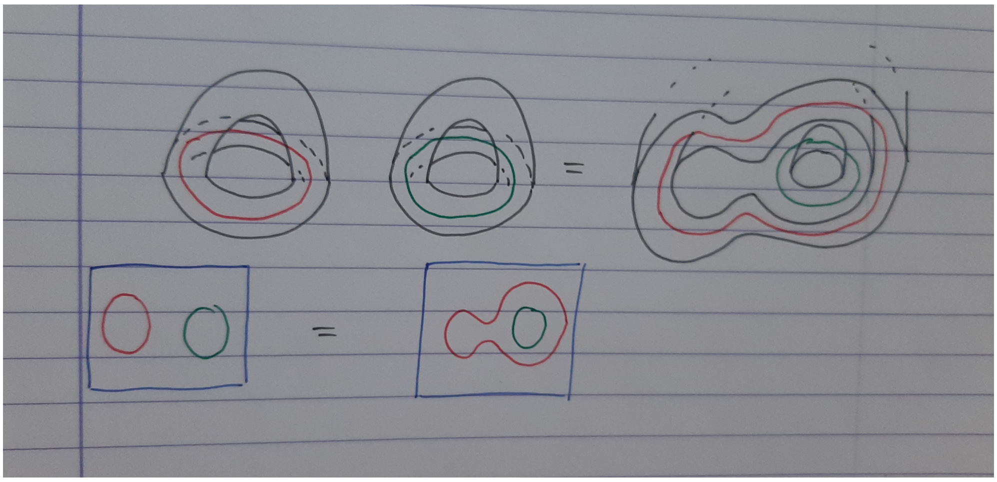


Figure 33: The handle slide move for a 2-handle

each generator in G. Then we attach 2-handles. Note that $\partial h_2 = S^1 \times D^2$, so the embedding $\alpha_i : S^1 \times D^2$ is determined up to isotopy by prescribing where the core of the solid torus is mapped to. Let this map be the path representing r_i in $\pi_1(X_1) \cong F\langle g_1, \ldots, g_n \rangle$. Then we are left with a manifold with boundary, so we close off with 3-handles and a single 4-handle. (This is possible by the Laudenbach-Poenaru theorem mentioned later in the course) Thus, we obtain a closed 4-manifold X which has $\pi_1(X) \cong G$.

13 Lecture 7 (14/3)

One of the main goals of the course is to represent closed, connected, oriented 3-manifolds in a diagrammatic way. Attaching a 3-dimensional 3-handle amounts to specifying a pair of discs on the plane (whose boundaries are identified in a orientation-reversing way), and to attach a 2-handle all we have to specify is a simple closed curve on that diagram. Hence we have a surjective map

$$\left\{ \begin{array}{c} \text{planar Heegaard} \\ \text{diagrams} \end{array} \right\} \twoheadrightarrow \left\{ \begin{array}{c} \text{closed, connected,} \\ \text{oriented 3-manifolds} \end{array} \right\},$$

where by a planar Heegaard diagram we mean one such that the the boundary of the union of 0,1 and 2-handles is S^2 This amounts to saying that the complement of the attaching curves for 2-handles on the diagram is connected. According to the Cerf theorem, any two diagrams encoding the same manifold must be related by cancelling pairs of handles and handle slides, that is

$$\frac{\left\{ \begin{array}{c} \text{planar Heegaard} \\ \text{diagrams} \end{array} \right\}}{\begin{array}{c} \text{stabilisation} \\ \text{handle slide} \\ \text{planar isotopy} \end{array}} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{closed, connected,} \\ \text{oriented 3-manifolds} \end{array} \right\}$$

is a bijection. On the other hand we have also studied another description of 3-manifolds, namely performing surgery along a framed knot. We have then a commutative diagram

$$\begin{array}{ccc} \{ \text{ framed links in } S^3 \} & \xrightarrow{\text{surgery}} & \left\{ \begin{array}{c} \text{closed, connected,} \\ \text{oriented 3-manifolds} \end{array} \right\} \\ & \searrow \quad\quad \nearrow_{\partial} & \\ & \left\{ \begin{array}{c} \text{4-dimensional} \\ \text{2-handlebodies} \end{array} \right\} & \end{array}$$

where the horizontal arrow is surjective by the Lickorish-Wallace theorem (hence the right bottom map). A 4-dimensional *2-handlebody* is a compact, connected, oriented 4-manifold of the form $h_0 \cup (\bigcup_i h_2)$. Given a framed link L, the left bottom arrow attaches $\#L$ 2-handles to D^4 along the components of L.

The Cerf theorem tells us how the 2-handles of two diffeomorphic 2-handlebodies are related, namely by handle slides. So we can mod out the 2-handles slides to get an injection

$$\frac{\left\{ \begin{array}{c} \text{framed links} \\ \text{in } S^3 \end{array} \right\}}{\text{2-handle slide}} \hookrightarrow \left\{ \begin{array}{c} \text{4-dimensional} \\ \text{2-handlebodies} \end{array} \right\}$$

but the surgery map does not become an injection (if in a composite of two maps the first is injective, the composite might NOT be injective (!)). In this case, $D^4 = h_0$ and $\mathbb{CP}^2 - int(D^4)$ have S^3 as boundary whereas the two 4-manifolds are not diffeomorphic. $\pm\mathbb{CP}^2 - int(D^4)$ arises from attaching a 2-handle determined by the unknot with framing ± 1.

It turns out that, together with handle slides, the addition or deletion of a unknot component with framing ± 1 (called *blow-up* or a *blow-down*, respectively) is enough to relate two knots whose surgery determines the same surgery.

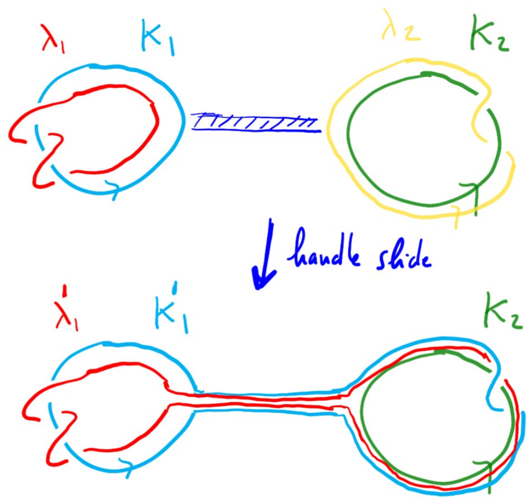
Theorem 13.1. *[Kirby 70s] Surgery defines a bijection*

$$\frac{\{\textit{ framed links in } S^3 \ \}}{\substack{\textit{blow-up/down} \\ \textit{handle slide}}} \xrightarrow{\ \cong\ } \left\{ \begin{array}{c} \textit{closed, connected,} \\ \textit{oriented 3-manifolds} \end{array} \right\}$$

That is, given two framed links L, L' in S^3, surgery on them defines diffeomorphic manifolds if and only if there is a sequence of blow-up/down 's and handle slides turning L into L' (as framed links).

We now proceed to explain what a handle slide looks like for a framed link: given two components K_1, K_2 of an oriented framed link (they could possibly be knotted), let λ_1, λ_2 be the longitudes defining each framing. Then a handle slide defines $K_1 \cup K_2$ by $K_1' \cup K_2$ where $K_1' := K_1 \#_b \pm \lambda_2$. Here b is some band connecting K_1 and λ_2. The sign $\pm$ is forced by the choice of the band, as it forces the orientation of the longitude.

What about the framings? The component K_2 keeps its framing λ_2, whereas K_1' gets as framing the following curve: $\lambda_1' = \lambda_1 \#_b \pm par(K_2)$, where $par(K_2)$ stands for a parallel copy of K_2.

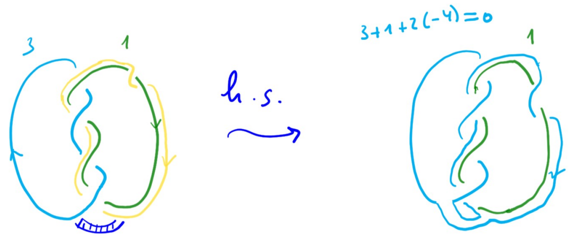


Exercise 13.2. Check that, viewing the framings as integers, if n_i is the framing for K_i, then the framing n_1' for the new component K_1' is given by

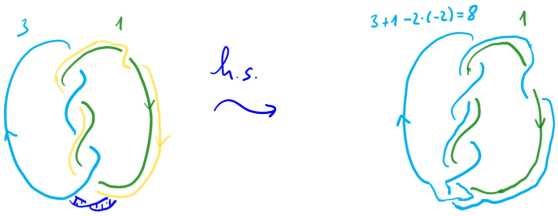
$$n_1' = n_1 + n_2 \pm 2\ lk(K_1, K_2),$$

where the sign $\pm$ is the same as the one in $K_1 \#_b \pm \lambda_2$.

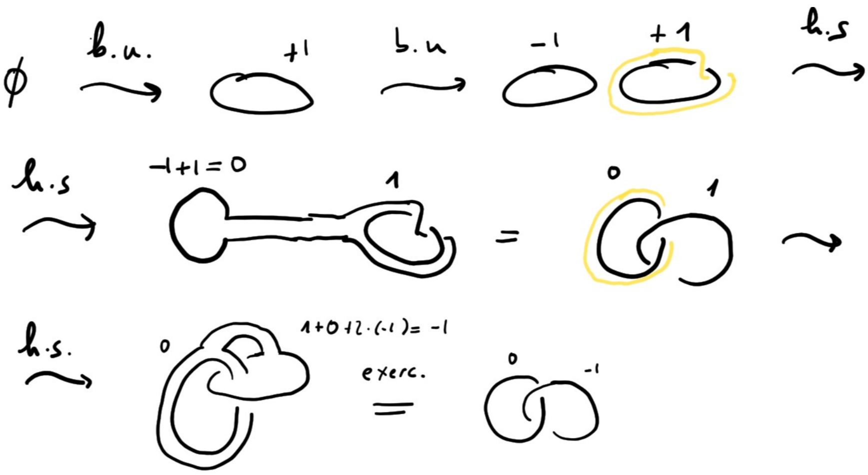
Example 13.3. The following is a handle slide:



Observe that the choice of band can dramatically change the handle slide:



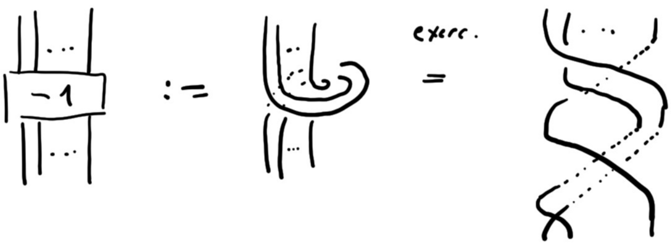
Example 13.4. The following framed links represent all S^3:



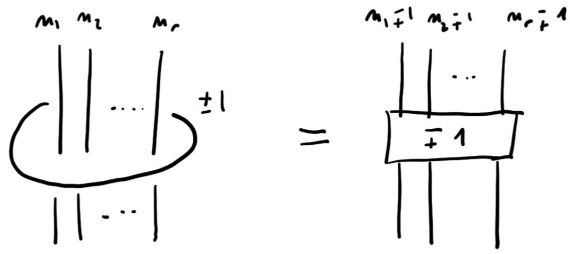
Given a set of r strands, as below,

write a box with $+1$ for a full right twist,

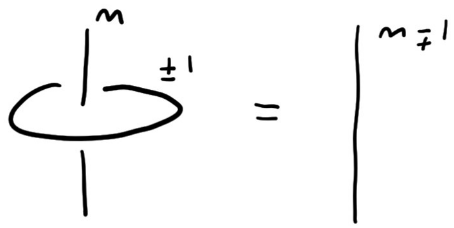
and a box with -1 for a full left twist,



Proposition 13.5. *Suppose that r parallel strands belonging to different components pierce the disc bounding an unknot with framing ± 1 exactly once. Then changing a framed link diagram as indicated locally produces diffeomorphic manifolds:*

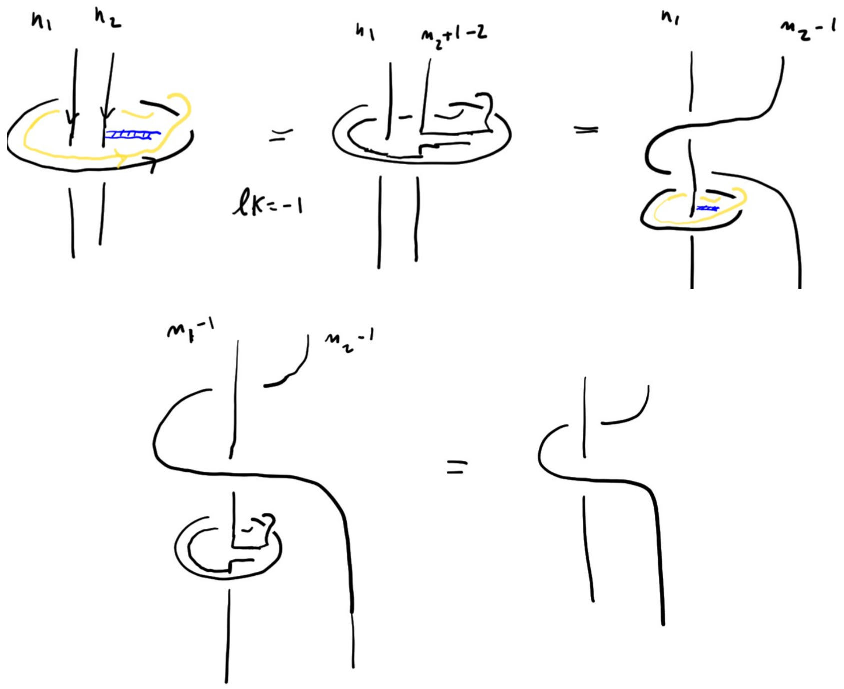


For the sake of concreteness, let us depic what this equality looks like for $r = 1$



and $r = 2$

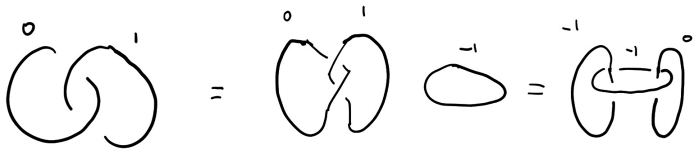
Proof (of the proposition). We just especify the case $r = 2$, the general case follows iterating the argument.



□

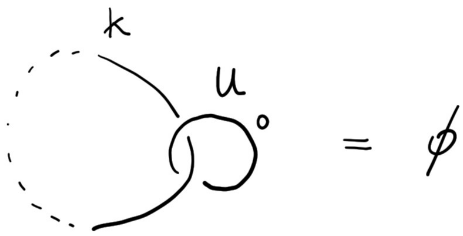
Example 13.6. The following framed links represent all the same manifold:

Example 13.7. The following framed links represent all the same manifold:

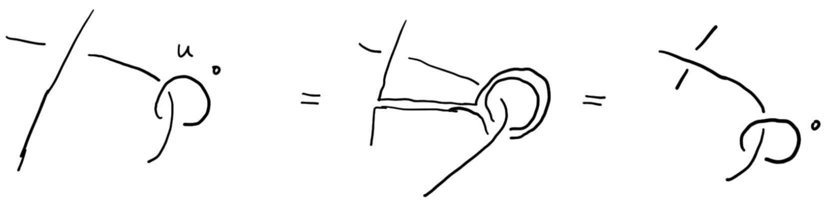


Exercise 13.8. Show that any knot K can be turned into the unknot by changing the sign of some of its crossings.

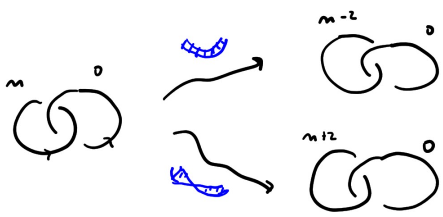
Proposition 13.9. *Let $L = K \cup U$ be the 2-component link formed by a knot K and an unknot component U with framing 0 with the property that K pierces the disc bounding U exactly once. Then surgery on L produces S^3.*



Proof. The sign of any crossing of K can be changed by a handle slide as follows:

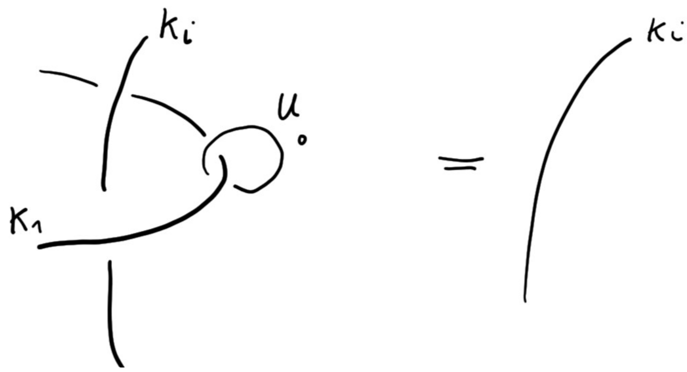


Applying this argument a number of times, by the previous exercise, we obtain a Hopf link with one component 0-framed and the other n-framed for some integer n. By handle sliding the n-framed knot over the 0-framed unknot, we can change the framing by ± 2. We hence get that the link is equivalent to the Hopf link where one component is 0-framed and the other is either 0-framed or 1-framed. In both cases surgery on these links produces S^3.



□

Corollary 13.10. *Given a framed link $L = U \cup K_1 \cup K_2 \cup \cdots$ where U is the unknot with framing 0 with K_1 piercing the disc bounding U exactly once (and not any other component), then surgery on L produces the same 3-manifold as the link obtained from L by removing the components U and K_1.*



Proof. Exercise. □

14 $L(p, 1)$ as the boundary of a 4-manifold (Ruben van Dijk)

In Section 8 we introduced lens spaces as quotients under a certain group action, and subsequently found an equivalent description using Dehn surgery in Theorem 8.7. Although the surgical description is more abstract, it allows us to generalise the definition of $L(p, q)$ by dropping the requirement that p and q are positive. Henceforth, we will therefore use this as our definition of a lens space:

Definition 14.1 (Lens spaces, revisited)**.** Let $p, q \in \mathbb{Z}$ be coprime integers. The lens space $L(p, q)$ is the $-p/q$-surgery on the unknot in S^3.

Explicitly, this means that

$$L(p, q) = (S^1 \times D^2) \cup_h (S^1 \times D^2),$$

where $h\colon S^1 \times S^2 \to S^1 \times S^2$ is a diffeomorphism that sends the longitude of one torus to the longitude of the other, and the meridian to a curve that runs p times along (same direction) its longitude and q times against (opposite direction) its meridian.

J: S^2 or S^1?

Example 14.2. We already saw in the first homework that $L(0, 1) \cong S^2 \times S^1$.

Exercise 14.3. Allowing p and q to be negative gives rise to many redundancies. Show that the following lens spaces are diffeomorphic:

$$L(p, q) \cong L(p, -q) \cong L(-p, q) \cong L(-p, -q).$$

Which of these diffeomorphisms preserve orientation? Do Propositions 8.8, 8.9 and 8.10 still hold now that p and q need not be positive?

Now, consider the lens space $L(p, 1)$ and the corresponding diffeomorphism h. Define a 4-manifold consisting of two 2-handles by

J: Isn't there any 0-handle?

$$E_p := (D^2 \times D^2) \cup_H (D^2 \times D^2),$$

where $H\colon S^1 \times D^2 \to S^1 \times D^2$ extends h by sending $S^1 \times \{0\}$ to itself and the meridian on the boundary $\partial(S^1 \times D^2) = S^1 \times S^1$ of one solid torus to a curve on the other's, which runs p times along its longitude but only once against its meridian.

Then, since

$$E_p \cong (S^3 \times [0, 1]) \cup_H (D^2 \times D^2),$$

Theorem [[theorem from lecture 10 March]] tells us that the boundary of E_p is given by $\partial E_p = L(p, 1)$.

J: Do you mean $D^3 \times [0, 1]$?

Exercise 14.4. Give an expression for H in terms of h, using Alexander's Lemma.

Example 14.5. The manifold E_0 can be described as a trivial bundle over S^2 with fiber D^2, so $E_0 \cong S^2 \times D^2$. Its boundary is $\partial E_0 \cong S^2 \times S^1 \cong L(0, 1)$.

Example 14.6. Consider the complex projective plane $\mathbb{CP}$, its charts

$$U_i := \{(z_0 : z_1 : z_2) : z_i \neq 0\},$$

and the corresponding diffeomorphisms $\varphi_i\colon U_i \to \mathbb{C}^2$ defined by $\varphi_0(z_0 : z_1 : z_2) := (z_1/z_0, z_2/z_0)$, $\varphi_1(z_0 : z_1 : z_2) := (z_0/z_1, z_2/z_1)$, and $\varphi_2(z_0 : z_1 : z_2) := (z_0/z_2, z_1/z_2)$. Notice that

$$U_0 \cup U_1 = \mathbb{CP} \backslash \{(0 : 0 : 1)\},$$

so

$$\mathbb{C}^2 \cup_\Phi \mathbb{C}^2 = \varphi_0(U_0) \cup_\Phi \varphi_1(U_1) \cong \mathbb{CP}\backslash\{(0:0:1)\},$$

where Φ glues the images of the intersection of domains $\varphi_0(U_0 \cap U_1) = \varphi_1(U_0 \cap U_1) = \mathbb{C}^* \times \mathbb{C}$. Explicitly, $\Phi(z,w) = \varphi_0 \circ \varphi_1^{-1}(z,w) = (z^{-1}, z^{-1}w)$.

J: Why?

This map sends all $(z,w) \in \mathbb{C}^* \times \mathbb{C}$ with $|z| \leqslant 1$ to elements (z', w') with $|z'| \geqslant 1$ and vice versa, so we can restrict ourselves to gluing cylinders $D^2 \times \mathbb{C} \cong \{(z,w) \in \mathbb{C}^2 : |z| \leqslant 1\}$ along their boundaries $S^1 \times \mathbb{C} \cong \{(z,w) \in \mathbb{C}^* \times \mathbb{C} : |z| = 1\}$:

$$\mathbb{C}^2 \cup_\Phi \mathbb{C}^2 \cong (D^2 \times \mathbb{C}) \cup_{\Phi|_{S^1\times\mathbb{C}}} (D^2 \times \mathbb{C}).$$

We can restrict Φ even further by demanding that $|w| \leqslant 1$, so that we end up gluing two copies of $D^2 \times D^2$ along $S^1 \times D^2$:

$$(D^2 \times D^2) \cup_{\Phi|_{S^1\times D^2}} (D^2 \times D^2).$$

Now, recall that the gluing map sends (z,w) to $(z^{-1}, z^{-1}w)$; restricted to the solid torus, this sends $S^1 \times \{0\}$ to itself, and the meridian on the boundary to a curve that runs once along the longitude and once against the meridian. In other words, $\Phi|_{S^1\times D^2}$ is the diffeomorphism we previously called H corresponding to $p = 1$, so

$$E_1 = (D^2 \times D^2) \cup_{\Phi|_{S^1\times D^2}} (D^2 \times D^2).$$

However, the last restriction we made causes this manifold to no longer be diffeomorphic to the punctured projective plane $\mathbb{CP}\backslash\{(0:0:1)\}$. To resolve this, consider the interior of the disk $D^4 \subset \mathbb{C}^2$, without its center $\varphi_2(0:0:1) = (0,0)$. Its preimage under φ_2 is

$$A := \varphi_2^{-1}(\operatorname{int} D^4\backslash\{(0,0)\}) = \{(z:w:1) \in \mathbb{CP} : |z|, |w| \in (0,1)\},$$

and hence $A \subset U_0$ and $A \subset U_1$. The image of A under φ_0 is

$$\varphi_0(A) = \{\varphi_0(z:w:1) : |z|,|w| \in (0,1)\} = \{(w/z, 1/z) : |z|,|w| \in (0,1)\} = \mathbb{C} \times (\mathbb{C}\backslash \operatorname{int} D^2),$$

and similarly

$$\varphi_1(A) = \mathbb{C} \times (\mathbb{C}\backslash \operatorname{int} D^2).$$

Now,

$$\begin{aligned} \mathbb{CP}\backslash \operatorname{int} D^4 &= (\mathbb{CP}\backslash\{(0:0:1)\})\backslash A \\ &\cong \varphi_0(U_0\backslash A) \cup_{\Phi|_{\mathbb{C}^*\times D^2}} \varphi_1(U_1\backslash A) \\ &= (\mathbb{C} \times D^2) \cup_{\Phi|_{\mathbb{C}^*\times D^2}} (\mathbb{C} \times D^2) \\ &\cong (D^2 \times D^2) \cup_{\Phi|_{S^1\times D^2}} (D^2 \times D^2), \end{aligned}$$

and thus

$$E_1 \cong \mathbb{CP}\backslash \operatorname{int} D^4.$$

Since $\mathbb{CP}$ is a complex manifold, it has a canonical orientation induced by the usual orientation of a complex vector space. It turns out that the above diffeomorphism reverses this orientation, and hence the diffeomorphisms

$$E_1 \cong -\mathbb{CP}\backslash \operatorname{int} D^4$$

and

$$E_{-1} \cong \mathbb{CP}\backslash \operatorname{int} D^4$$

are orientation-preserving.

15 Lecture 8 (21/3)

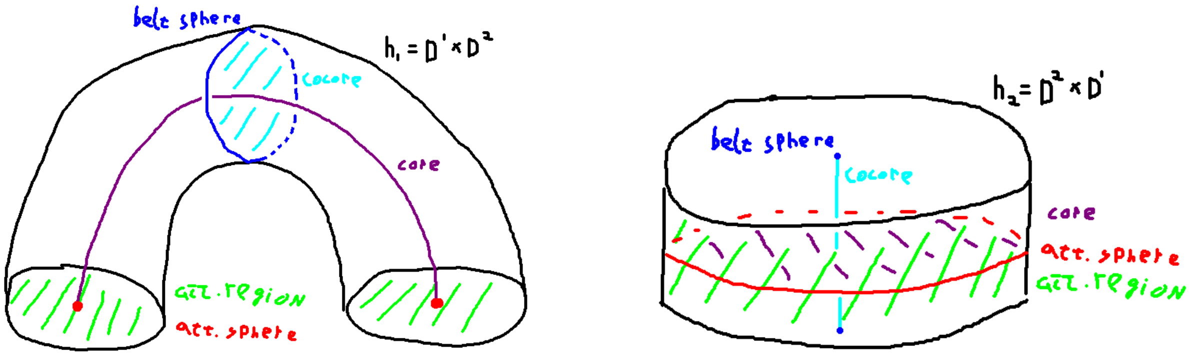
Thusfar we have learned to describe (handle decompositions of) manifolds in two and three dimensions: surfaces can be represented as Kirby diagrams on the line, and 3-manifolds as Heegaard diagrams on the plane. In this section, we set out to treat 4-manifolds in a similar manner, as diagrams in 3-space, after recalling some terminology about handle decomposition.

The geometry of handles

Let $h_k = D^k \times D^{n-k}$ be a k-handle. We can distinguish the following subsets of h_k:

Attaching region	$\partial D^k \times D^{n-k}$
Attaching sphere	$\partial D^k \times \{0\}$
Core	$D^k \times \{0\}$
Cocore	$\{0\} \times D^{n-k}$
Belt sphere	$\{0\} \times \partial D^{n-k}$

For $n = 3$, these parts of h_1 and h_2 are shown in Figures 34a and 34b.



(a) A 1-handle.

(b) A 2-handle.

Figure 34: Handles in 3-space.

In Kirby diagrams for surfaces, we drew a handle decomposition by indicating the attaching spheres of 1-handles on the line representing the boundary of the 0-handle $\partial D^2 \cong \mathbb{R} \cup \{\infty\}$; the attachment of the single 2-handle was implicit.

In Heegaard diagrams for 3-manifolds, the boundary of the 0-handle $\partial D^3 \cong \mathbb{R}^2 \cup \{\infty\}$ was represented by the plane. The 1-handles were indicated by their attaching regions and the 2-handles by their attaching spheres; the one 3-handle was once again implicit.

We also saw that we can obtain different handle decompositions (and hence different diagrams) for the same manifold via handle moves. One such move is handle cancellation:

Proposition 15.1. *A k-handle h_k and $(k+1)$-handle h_{k+1} can be cancelled if the belt sphere of h_k intersects the attaching sphere of h_{k+1} at a single point.*

Kirby diagrams for 4-manifolds

In 4-space, a 0-handle is a ball, with boundary homeomorphic to $\mathbb{R}^3 \cup \{\infty\}$. We can thus use 3-space to represent the boundary of this handle. To this boundary, we attach 1-handles $D^1 \times D^3$ along their attaching region $\partial D^1 \times D^3$, i.e. the disjoint union of two 3-balls. Therefore, a 1-handle for a 4-manifold can be represented as a pair of balls in 3-space.

The union (0-handle) $\cup$ (1-handle) has boundary $S^1 \times S^2$; this can be shown by shifting one ball in the corresponding diagram to include the point at infinity. More generally, the union of

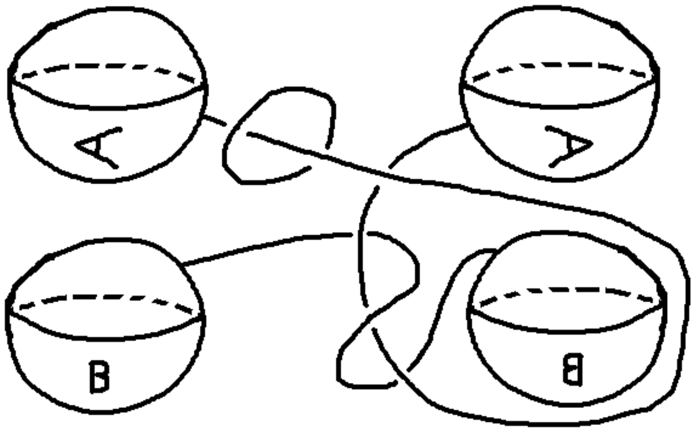


Figure 35: A Kirby diagram for a manifold with two 1-handles and three 2-handles.

a 0-handle and g 1-handles has boundary

$$\partial\left(\natural_g S^1 \times D^3\right) = \#_g S^1 \times S^2.$$

Now, 2-handles are represented by their attaching spheres on this boundary, depicted as framed links and framed arcs between 1-handles, as in Figure 35.

Describing the 1- and 2-handles is in fact enough to describe the entire 4-manifold, as we will now show.

Assuming that the manifold in question is closed and connected, we can find a handle decomposition that has a single 4-handle. Via duality, the union of this 4-handle and g 3-handles is diffeomorphic to the union of a 0-handle and g 1-handles, and hence its boundary is diffeomorphic to $\#_g S^1 \times S^2$. Hence, for our manifold to be closed, we must have that $X_1 :=$ (0-handle) $\cup$ (1-handles) $\cup$ (2-handles) has boundary $\partial X_1 = \#_g S^1 \times S^2$.

The Laudenbach–Poénaru Theorem says that any diffeomorphism from $\#_g S^1 \times S^2$ to itself can be extended to a self-diffeomorphism on $\natural_g S^1 \times D^3$, so if $\partial X_1 = \#_g S^1 \times S^2$, there is a unique closed manifold that can be obtained by attaching 3-handles and a 4-handle to X_1. In other words, a closed 4-manifold is completely described by its 1- and 2-handles.

16 Torus knots (Ruben IJpma)

Given any knot $K : S^1 \hookrightarrow \mathbb{R}^3$, computing the fundamental group $\pi_1(R^3 \backslash K)$ is of interest, since it allows one to distinguish different types of knots. (We have the convention of denoting by K both the embedding and its image set.) The specific purpose of this lecture is to elaborate on Example 3 given earlier, where the case of torus knots is briefly discussed. To this end, first recall the following version of the Van Kampen theorem, which will be necessary for our computation.

Theorem 16.1. *Suppose that X is covered by path-connected open sets U, V, and that $U \cap V$ is path-conneceted. Let $x_0 \in U \cap V$, and suppose that*

$$\begin{aligned} \pi_1(U, x_0) &= \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle, \\ \pi_1(V, x_0) &= \langle h_1, \dots, h_p \mid s_1, \dots, s_q \rangle, \\ \pi_1(U \cap V, x_0) &= \langle f_1, \dots, f_i \mid q_1, \dots, q_j \rangle. \end{aligned}$$

Let $i^{U,V} : U \cap V \to U, V$ denote the inclusions. Van Kampen says that the inclusions $U, V \hookrightarrow X$ induce an isomorphism

$$\pi_1(X, x_0) \cong \langle g_1, \dots g_n, h_1, \dots h_n \mid r_1, \dots, r_m, s_1, \dots, s_m, i^U_*(f_k) = i^V_*(f_k), k = 1, \dots, i \rangle.$$

Let us now generalise Example 3 to include more types of torus knots. We take $S^1 = \{\theta \bmod 2\pi\}$, and $D^2 = \{(r, \theta) : r \in [0,1], \theta \in S^1\}/(\{0\} \times S^1 \sim *)$. Let T be the standard embedding of the torus $S^1 \times S^1 \hookrightarrow \mathbb{R}^3$ given by

$$(\varphi, \psi) \longmapsto ((2 + \cos(\psi))\cos(\varphi), (2 + \cos(\psi))\sin(\varphi), \sin(\psi)).$$

For $m, n > 0$ be with $\gcd(m, n) = 1$, define $K_{m,n} : S^1 \to \mathbb{R}^3$ to be the map $\theta \mapsto (m\theta, n\theta)$ composed with T. For $m = 1$ or $n = 1$ this gives the unknot. For $m, n > 1$, this gives a knot wrapping around the torus m times longitudinally and n times meridionally.

Exercise 16.2. Draw the image of $K_{i,j}$ for $i, j = 1, 2, 3$ from the top view of the torus. Figure 3 already gives a few of these. Realise that $K_{m,n}$ is not injective if $\gcd(m, n) > 1$.

To make matters easier, let $T = S^1 \times S^1$, and we view T to sit inside S^3 via the splitting $S^3 = \partial D^4 = (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2)$, which is made explicit by the pushout diagram

$$\begin{array}{ccc} T & \xrightarrow{(\varphi,\psi)\mapsto(\varphi,1,\psi)} & S^1 \times D^2 \\ {\scriptstyle (\varphi,\psi)\mapsto(\psi,1,\varphi)} \downarrow & & \downarrow {\scriptstyle \iota_m} \\ S^1 \times D^2 & \xrightarrow[\iota_n]{} & S^3. \end{array}$$

Let K be the image of $\theta \mapsto (m\theta, n\theta)$ in $T \hookrightarrow S^3$. So, instead of computing $\pi_1(\mathbb{R}^3 \backslash K_{m,n})$, we will compute $\pi_1(S^3 \backslash K_{m,n})$. Note that $\pi_1(S^n \backslash \iota(F)) = \pi_1(\mathbb{R}^n \backslash F)$ for any compact F by an application of Van Kampen to the one-point compactification of $\mathbb{R}^n$, but we shall not discuss the details here.

In the above diagram, let $\mathbf{T}_m$ denote the upper-right solid torus, and $\mathbf{T}_n$ denote the bottom-left solid torus, then the meridians of $\mathbf{T}_m$ correspond to the longitudes of $\mathbf{T}_n$ and vice versa. We show that $S^3 \backslash K$ deformation retracts onto a space $X_m \cup X_n$, so that $\pi_1(S^3 \backslash K) \cong \pi_1(X_m \cup X_n)$. Here, $X_{m,n} \subset \mathbf{T}_{m,n}$ are spaces such that

$$X_m \cup X_n \cong \frac{[-1,1] \times S^1}{\substack{(-1,\theta)\sim(-1,\theta+2\pi/m) \\ (1,\theta)\sim(1,\theta+2\pi/n)}}.$$

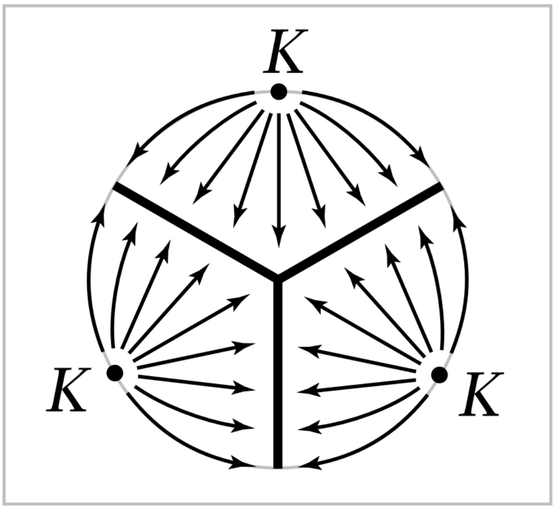


Figure 36: The sets $V_{m,\varphi}$ and $W_{m,\varphi}$, and the deformation retract $H_{m,\varphi}$.

Before describing the deformation retract, let us already compute the fundamental group. Let p denote the projection of the quotient space on the right hand side. Let $\varepsilon > 0$ be small, then the right hand side splits into $p((-\varepsilon, 1] \times S^1)$ and $p([-1, \varepsilon) \times S^1)$. It is straightforward to check that these images deformation retract onto $p(\{1\} \times S^1) = S^1 \bmod 2\pi n$ and $p(\{-1\} \times S^1) = S^1 \bmod 2\pi m$, respectively, and the latter spaces are both homeomorphic to S^1. Let γ be a loop generating the fundamental group of the intersection $p((-\varepsilon, 1] \times S^1) \cap p([-1, \varepsilon) \times S^1) \simeq S^1$. By the previous argument, γ is both a generator of the fundamental group of $p((-\varepsilon, 1] \times S^1)$ in which $\gamma \simeq \gamma^n$, and also a generator of the fundamental group of $p([-1, \varepsilon) \times S^1)$ in which $\gamma \simeq \gamma^m$. By Van Kampen, we finally get

J: Deformation retract onto $S^1 \bmod 2\pi n$? What do you mean?

J: Let op! This is a dangerous expression. Do you mean that the image of the generator of the 1-sphere in the intersection is n times the generator of $\pi_1 X_n$?

$$\pi_1(X_m \cup X_n) = \pi_1(p((-\varepsilon, 1] \times S^1) \cup p([-1, \varepsilon) \times S^1)) = \langle [\gamma] : [\gamma]^m = [\gamma]^n \rangle.$$

We get to the task of defining the space $X_m \subset T_m$. For any meridian $\{\varphi\} \times \partial D^2$ of $\mathbf{T}_m$, set

$$V_{m,\varphi} = \{\psi + n\varphi \in S^1 : m\psi = 0\}, \quad V'_{m,\varphi} = \{\psi + n\varphi + \pi/m \in S^1 : m\psi = 0\}$$
$$W_{m,\varphi} = [0,1] \times V'_{m,\varphi} \subset D^2.$$

Moving the $W_{m,\varphi}$ through φ traces out the 'corkscrew' $X_m = \bigcup_{\varphi \in S^1} \iota_m(\{\varphi\} \times W_{m,\varphi}) \subset \mathbf{T}_m$. Now, for each φ, there exists a deformation retract $H_{m,\varphi}$ of $D^2 \backslash V_{m,\varphi}$ onto $W_{m,\varphi}$, as shown in Figure 36. A formula for the homotopy is, for example,

J. I cannot understand what you mean with these expressions.

$$H_{m,\varphi}(r, \theta, t) = \left(r \cdot (1 - t \cdot (1 - r) \cdot \frac{m}{\pi} d(\theta, V'_{m,\varphi})), \quad \theta \cdot (1 + t \cdot (-1)^{\lfloor (\theta - n\varphi)m/\pi \rfloor} d(\theta, V'_{m,\varphi})) \right).$$

(Here, d denotes the infimal distance in S^1.) Taking the union over φ, we obtain a deformation retract H_m from $\mathbf{T}_m \backslash K$ onto X_m. The space $X_n \subset \mathbf{T}_n$ and deformation retract H_n follow the same construction with m and n switched.

Proposition 16.3. *We have that $X_m \cap T = X_n \cap T = X_m \cap X_n$ is a parallel copy of K. Furthermore, the map*

$$[0,1] \times S^1 \to \bigcup_{\varphi \in S^1} \{\varphi\} \times W_{m,\varphi}$$
$$(h, \varphi) \mapsto H_{m,\varphi}(h, \varphi, 1)$$

induces a homeomorphism $X_m \cong [0,1] \times S^1/(0, \theta) \sim (0, \theta + 2\pi/m)$, and likewise for X_n.

To define a deformation retract on $S^3 \backslash K$, we would need H_m and H_n to agree on $T \backslash K$, which they do not. This can be fixed by choosing constants α, β such that

$$H_m(\alpha\varphi, 1, \theta, t) = H_n(\beta\theta, 1, \varphi, t),$$

and redefining the homotopies accordingly. In this way, the flow is directed perpendicularly to the knot. Finally, this defines a deformation retract from $S^3 \backslash K$ onto $X_m \cup X_n$.

17 Equivalent surgery descriptions I (Maurits Brinkman)

In this section, we will show that the framed link given in Figure 37 is a surgery description for the Poincaré homology sphere. This can be seen as a particular example of the theory treated in Lecture 7 (14/3), where the main principle we use is Kirby's theorem: two framed links give a surgery description for the same manifold (up to isomorphism) if and only if those framed links differ by blow-ups, blow-downs and/or handle slides.

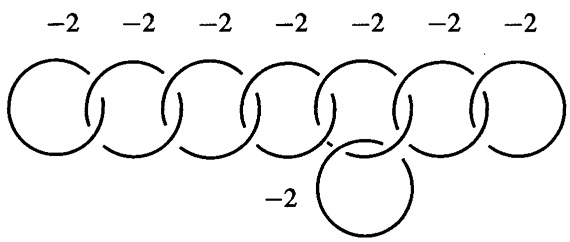


Figure 37: Framed link

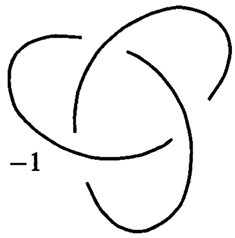


Figure 38: Left-handed trefoil

Now, as we know from Boudewijn's presentation on the Poincaré homology 3-sphere, we know that the Poincaré homology sphere is equivalent to the surgery description given by the trefoil knot having framing -1. Therefore, by Kirby's theorem, it is enough to show that the framed links in Figure 37 and Figure 38 are related by some sequence of blow-ups/blow-downs and handle slides.

In the first part of the sequence, we will use two types of moves. The first one, shown in Figure 39, is the consequence of a handle slide, as we did in Lecture 7 (14/3). The second one, a blow-down, where now two components pierce the disc of the unknot having framing -1. Notice that both are particular examples of Proposition 13.5.

To remark, the different colours that are used have no other purpose than clarification of which components are added, and which vanish during the process. Also, the equality signs represent an equivalence relation defined by the fact that the links which are considered equivalent produce the same manifold after surgery.

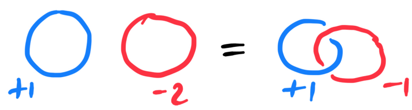


Figure 39: Consequence of a handle slide

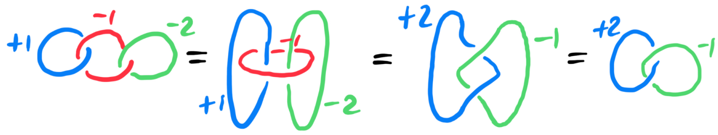


Figure 40: Consequence of Proposition 13.5

Now, in Figure 41 we start transforming the framed link of Figure 37. The first equality follows from a blow-up where we introduce three unknots having framing $+1$. By using a handle slide, in exactly the same way as in Figure 39, the unknots (we just introduced) of framing $+1$ are linked to the outer unknots having framing -2. The third, fourth, fifth and sixth equality in Figure 41 are repetitions of performing a blow-down on the component having framing -1, in the fashion of Figure 40.

Having ended with the bottom right-most framed link in Figure 41, one could think that we can still apply a blow-down using the component in green having framing $+1$... *But*, this time, we do not have two components piercing through the unknot of framing $+1$, but rather *three* components (the ones in blue). As we use Proposition 13.5 for $r = 3$. By doing this (first equality in Figure 42; the colours are different from the last framed link in Figure 41, for clarification purposes), one produces a full left twist of the three components piercing through the disc. After this, we perform a blow-down (second equality in Figure 42), which is possible

J: How do you get the middle link in figure 42 from the left one after this "full twist"?

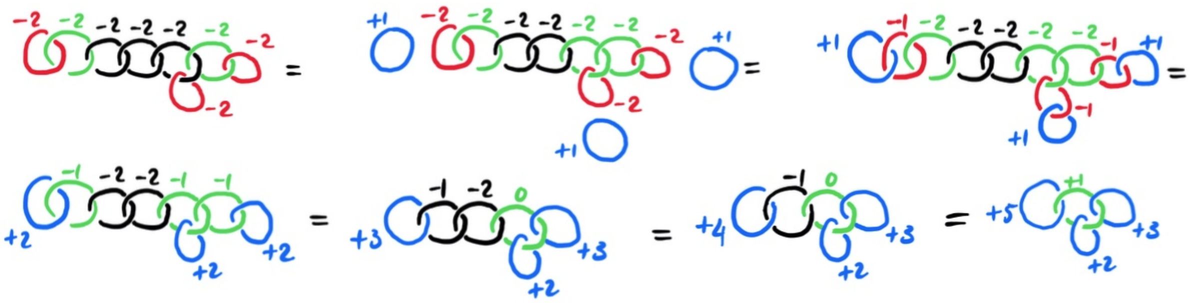


Figure 41: Performing blow-ups, blow-downs and handle slides on the framed link of Figure 37

since the two components pierce the disc of the unknot (the one in red; framing +1) once. After this, we end up with a framed link where the same component (in orange) pierces the unknot (in dark blue) twice. To further reduce this framed knot (the last one in Figure 42, we again apply Proposition 13.5, which gives another full left twist. The new framing can be calculated by the formula given in Exercise 13.2: $1 + 3 - 2 \cdot [\text{linking number}] = 1 + 3 - 2 \cdot 2 = 0$

J: Again, how do you get the right hand picture of figure 42?

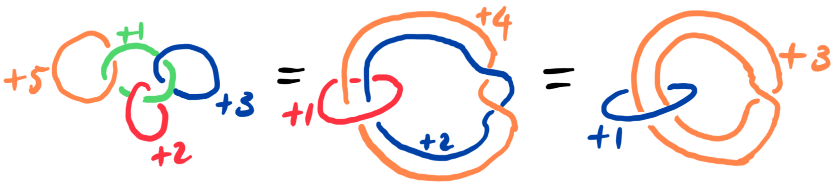


Figure 42: Performing a full left twist and blow-down, respectively

J:Not finished...

18 Equivalent surgery descriptions II (Bram Brongers)

The aim of this lecture will be to show the equivalence of two surgery diagrams. We use the techniques from lecture 12, in particular the handle slide move and the blow-up/blow-down construction. We refer to 13.1 for the details. The final goal is to show the equivalence of the two diagrams 43: Note that throughout all the images 44, 45, 46 and 48 we have marked the

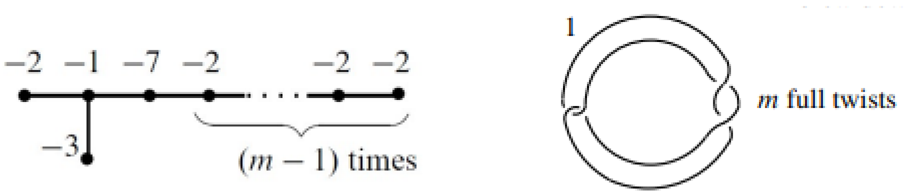


Figure 43: The desired equivalence.

equality signs with a number, which will correspond to the explanation below.

Remark 18.1. We will not worry about the orientations of the knots, since they are not displayed in the original text.

1. The starting point in the book is the graph which is on display in 44. By definition, this graph represents the link which is drawn. The final equivalence that we wish to demonstrate involves a total of m nodes with label -2, following the node labelled -7. However, we shall see that we can first consider this simpler case, and by the end of the procedure it will be completely clear how to obtain the general case.

2. Notice that the strands labelled -3, -2 and -7 pierce the disk with boundary -1 precisely once. Hence, we can blow this down. The result is that the three knots which pierced the disk now have linking number ± 1. We also need to shift the framing number, namely they all need to be increased by 1, since the knot we removed had framing number -1. This gives the second equality.

   J: Why is the resulting link precisely what you display? Is there any full twist?

3. Notice that the blow down has yielded another knot with framing number -1, and the corresponding disk is pierced by the strands labelled -2 and -6. Hence, we can again blow this down, at the cost of an addition twist to the remaining strands, and shifting their framing numbers up by 1, as above. This result is the third equality.

   J: Again it doesn't seem obvious to me that the link you get is that one precisely.

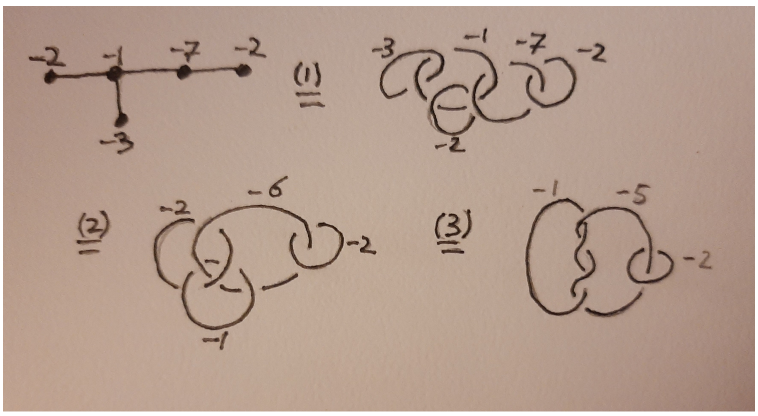


Figure 44: The first steps.

4. Now, we will neglect the knot with framing number -2 and focus on a way to rewrite the "interesting" part of the link which remains. In particular, we are going to perform the handle slide move on it. The setup for this is drawn at the start of 45. The red knot is a

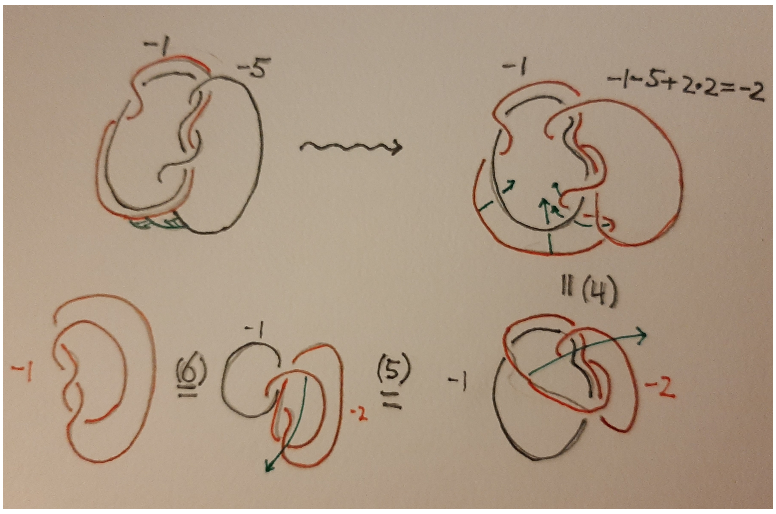


Figure 45: Performing the handle slide.

framing for the component labelled -1, and it is connected to the component labelled -5 by a green band. This results in the link drawn. Observe that the linking number of the two components is ± 2. Hence, the new framing number of the altered component becomes $-1-5\mp 2\cdot 2=2$, since we are using an orientation reversing band. We have picked 2 as the framing number because it makes the rest of the problem work. Then, we have drawn green arrows to indicate where we are pulling the red strand. After pulling it as indicated, we obtain the fourth equality.

5. The fifth equality simply results from pulling a strand over to the other side.

6. Now we notice that we have a link component with framing number -1, which is pierced by the other component only once. Hence, we can blow down again. Since there is only one strand piercing, this strand does not obtain an extra twist, but we do increase the framing number by 1. After pulling the only strand which remains, i.e. the red strand, we obtain the sixth equality.

7. We continue in 46. The seventh equality is a simply isotopy, as indicated by the green

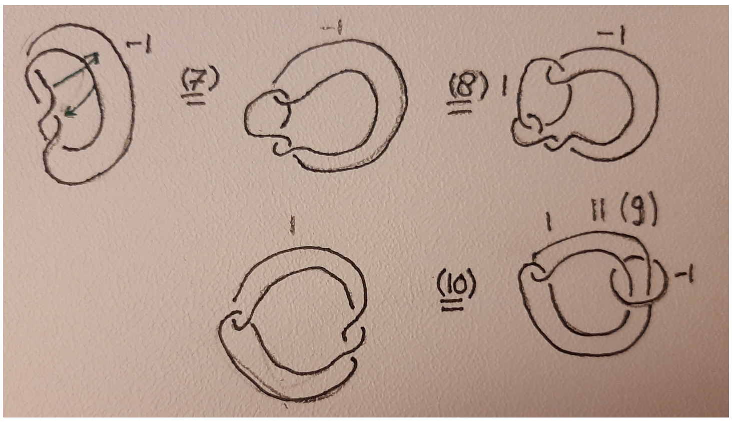


Figure 46: Rearranging the knot.

arrows.

8. We now notice that we have a self-intersection of our knot. We want to blow this up, to get the eighth equality. Strictly speaking, we cannot do this because both strands belong to the same knot. If we pretend this is not a problem, then performing the handle slide move on the resulting link gives us the same scenario as equality 6) - which means there indeed was not a problem. Thus, the blow up of the self-intersection results in the eigth equality.

J: But it is, especially for the framing. Why is the framing still -1 after introducing the new component?

9. This is an isotopy, the goal of which is to "interchange the roles" of the -1 and $+1$ components of the link. That is, we want to be able to blow down the -1 component, so we wish to drawn it as a little circle.

10. We are now nearly finished. The tenth equality is the result of the blow down, which we have to do using the handle slide again, for the same reason as before - the two strands piercing the disk belong to the same knot. This is resolved in the same manner as before, and the resulting knot is the tenth equality. Notice that we have obtained a twist.

J: What about the new framing?

11. The final result is as follows. We had ignored all of the -2 components of the link, up until now. If we had not, then they would still be linked to the -1 component, before the tenth equality. After the blow down move, the nearest -2 component which pierced the disk associated to the -1 component would obtain a new framing number, namely $-2+1=-1$. This would result in 47. Thus, we can blow this down again, at the cost of

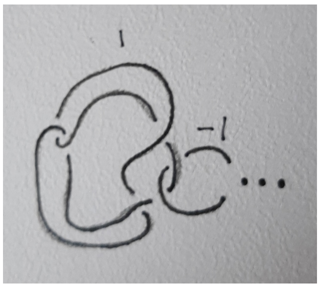


Figure 47: The general case.

an additional twist. Hence, if we have m-many -2 components added, we get m additional twists. This proves the result:

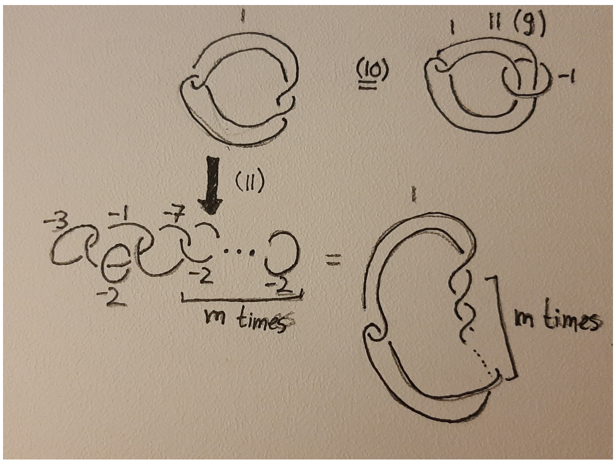


Figure 48: The desired equivalence.

19 Equivalent Kirby diagrams (Lisanne)

In this section, we will show the equivalence of the following Kirby diagrams:

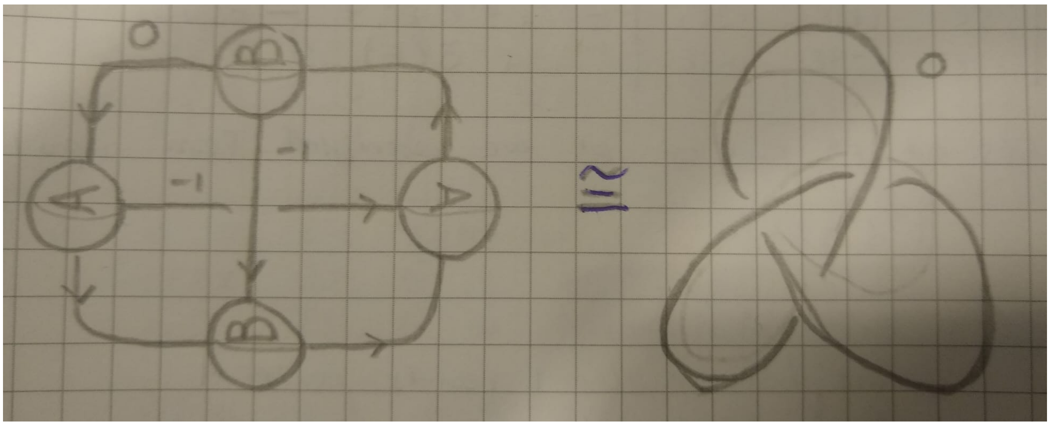


Figure 49: The desired equivalence

On the left we see a Kirby diagram with two 1-handles, three 2-handles, of which one with framing 0 and two with framing -1. From now on they will be called the round, horizontal and vertical 2-handle respectively. On the right we see a right-handed trefoil knot with framing 0.

We can split this process up into three parts

1. Cancellation of the horizontal 1-handle.

2. Cancellation of the vertical 1-handle.

3. Transforming the result from steps 1 and 2 into the trefoil knot.

In the first two steps we will be using 2-handle slides to ensure that we can cancel the "A" 1-handle. Then Doing handle slides, we need to keep the following things in mind.

1.

2.

3.

Part 1: Cancellation of horizontal 1-handle

In this part our goal is to cancel the horizontal 1-handle, from now on called the horizontal 1-handle using the ideas stated above. Note that I did not write the A's and B's and the framing everywhere, since they did not change from the starting figure and it would have made the pictures less clear.

Our first step is doing a handle slide of the round 2-handle w.r.t. the horizontal 2-handle with an untwisted band. This results in the following figure: As you can see, when doing a handle slide, we are combining the first two steps from above. Moreover, as you can see, the framing of the round 2-handle changed from 0 to -1. Using the formula above we see

J: To what are you glueing the band?

$$n_1' = 0 + (-1) + lk(L_1, L_2),$$

where L_1, L_2 are the original round and horizontal 2-handle respectively, which have linking number 0. So indeed, the new framing of the round 2-handle is -1.

Next we do another handle slide of the round 2-handle w.r.t the horizontal 2-handle, but now on the bottom and with twisted band. This results in the following figure: Again, following

J: Why does it have to be twisted?

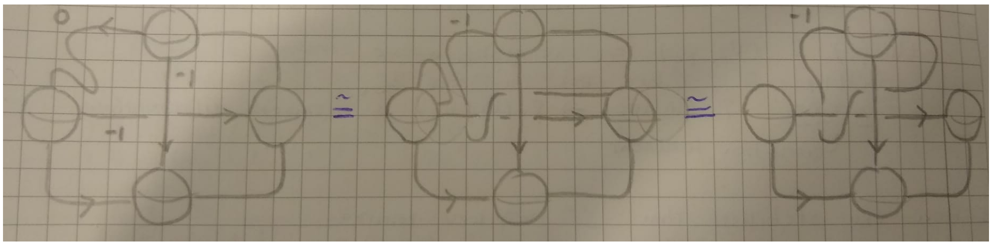


Figure 50: The first handle slide

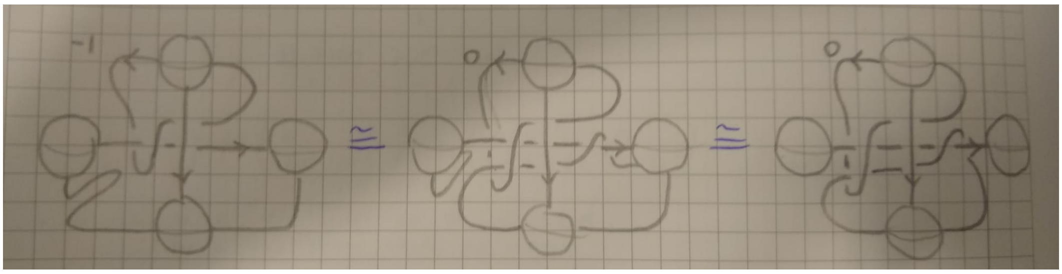


Figure 51: The second handle slide

orientation, the round 2-handle first goes under and then over the horizontal 2-handle. Moreover, it also entangles itself on the left side in the same way it did the horizontal 2-handle in the last step. The framing of the round 2-handle changed back to 0 again. Using the formula we see

$$n_1'' = -1 + (-1) - lk(L_1', L_2),$$

where L_1', L_2 are the round 2-handle from the last step and the horizontal 2-handle respectively. As logic entails, they have linking number -1, so we find

$$n_1'' = -1 - 1 - 2(-1) = -2 + 2 = 0.$$

After these two steps, the horizontal 2-handle is not connected to the rest of the Kirby diagram anymore, so by isotopy we can pull it out and then we can cancel it. This leaves us with the following Kirby diagram

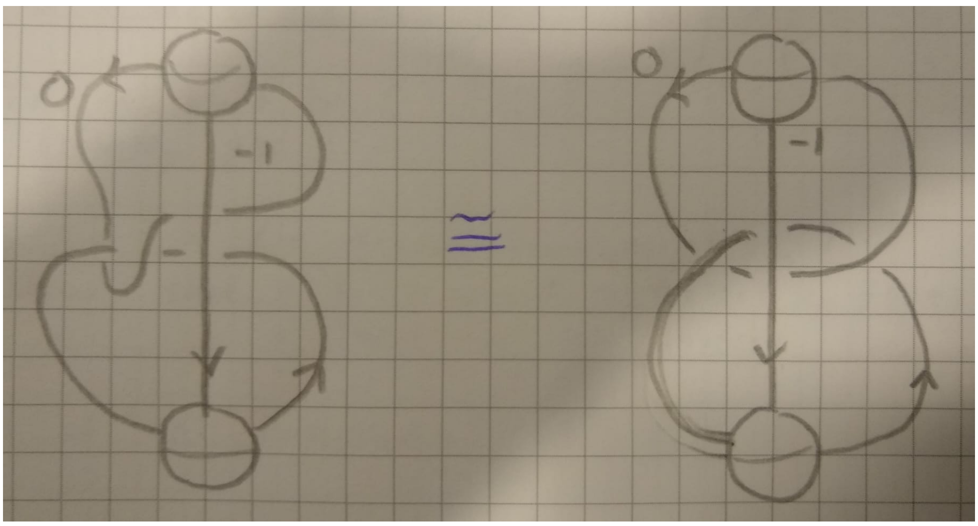


Figure 52: The resulting Kirby diagram before and after isotopy

Part 2: Cancellation of the vertical 1-handle

So, now we are left with a single 1-handle and two 2-handles, the round and vertical one. The idea of this part is practically the same as part 1, but everything is a bit more entangled already. Again, not everything is denoted in the picture because of the reasons above.

So, the first handle slide is of the round 2-handle w.r.t. the vertical 2-handle with an untwisted band. This results in the following figure: As you can see, the round 2-handle crosses

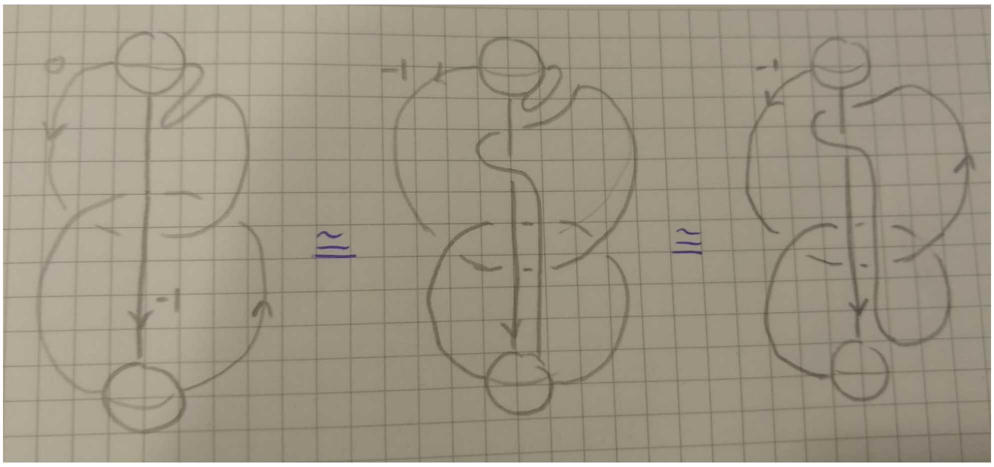


Figure 53: The third handle slide

over itself twice in the middle, since it follows the vertical 2-handle in everything it does. The framing of the round 2-handle changes again from 0 to -1, using the same logic as in step 1, where indeed the 2 links as seen Figure **??** have linking number 0.

Then doing another handle slide on the left with a twised band, we get: The framing of the

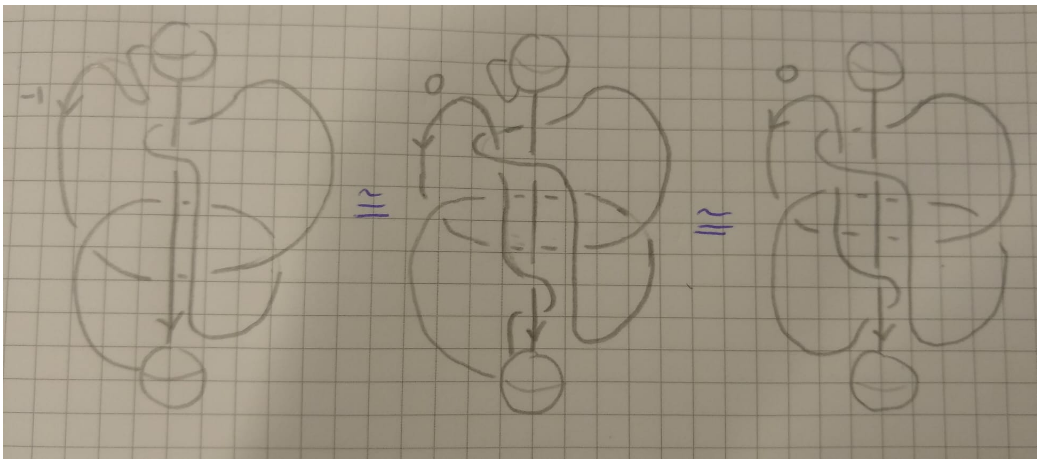


Figure 54: The fourth handle slide

round 2-handle changed back to 0 again as we desire via the same construction as in part 1. Now the vertical 1-handle is not connected anymore, so we can pull it out by isotopy and cancel it. This then leaves us with the following Kirby diagram:

Part 3: Transforming to the trefoil knot

We are now left with only a single 2-handle with framing 0, so we are on the right track. However, the Kirby diagram shown in Figure 55 does not really resemble the trefoil knot at all.

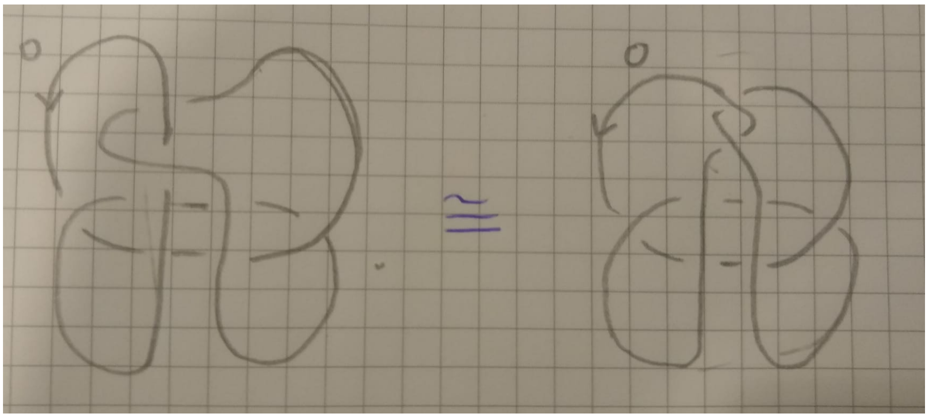


Figure 55: The resulting Kirby diagram before and after isotopy

To transform the resulting Kirby diagram into the trefoil knot, we will be using Reidemeister moves as seen in Figure 1.