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Homework 2 - Topics in Topology

March 18, 2022

Please return before April 8, 2022.

1. Argue whether the following statements are true or false (that is, either prove or give a counterexample):

   (a) Given a handle decomposition of some n-manifold, the number of 0-handles and the number of n-handles must always be equal to one.

   (b) If $n \geq 2$, there are always exactly two different ways to attach a 1-handle to D^n.

   (c) It is always possible to attach a 2-handle to any 3-manifold.

   (d) In a handle decomposition of a closed, connected, oriented 3-manifold with one 0-handle and one 3-handle, the number of 1-handles must be equal to the number of 2-handles.

   (e) In a handle decomposition of a closed, connected, oriented 4-manifold with one 0-handle and one 4-handle, the number of 1-handles must be equal to the number of 3-handles.

   (f) Any closed, connected, oriented 3-manifold is the boundary of a closed, connected, oriented 4-manifold.

2. Give a topological space whose fundamental group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$ (and show that this is indeed its fundamental group).

3. Let us write $T^2 := S^1 \times S^1$.

   (a) Show that the surfaces $T^2 \# \mathbb{RP}^2$ and $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$ are diffeomorphic.

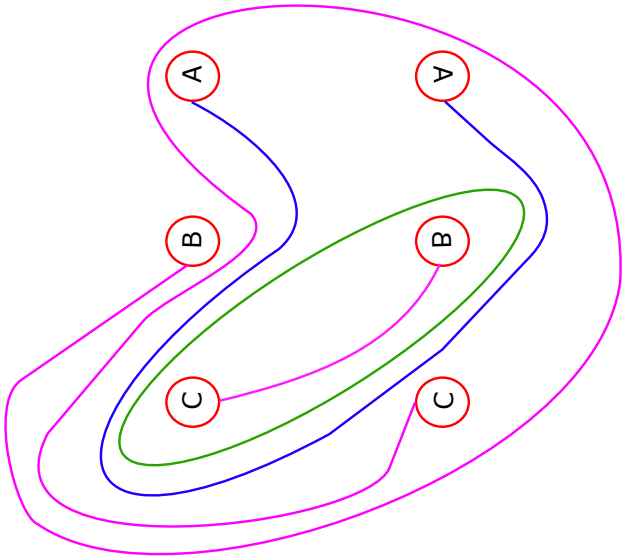
   (b) Let $\Sigma_2 := T^2 \# T^2$ be the genus 2 surface, and let $\Sigma_2 = h_0 \cup (\bigcup_{i=1}^4 h_1^i) \cup h_2$ be its standard handle decomposition. Show that $\Sigma_2 - int(h_2)$ is homotopy equivalent to a bouquet $S^1 \vee S^1 \vee S^1 \vee S^1$ of four 1-spheres.

   (c) Show that the fundamental group of Σ_2 is given by

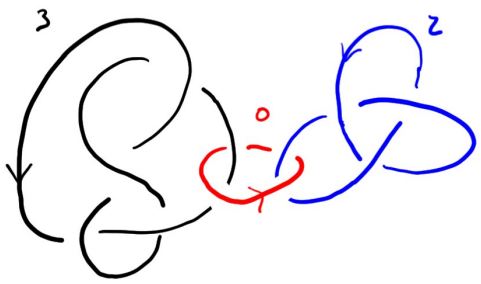
   $$\pi_1(\Sigma_2) \cong \frac{F(x_1, x_2, x_3, x_4)}{\langle x_1 x_2 x_1^{-1} x_2^{-1} x_3 x_4 x_3^{-1} x_4^{-1} \rangle}.$$

   What is the abelianisation of this group?

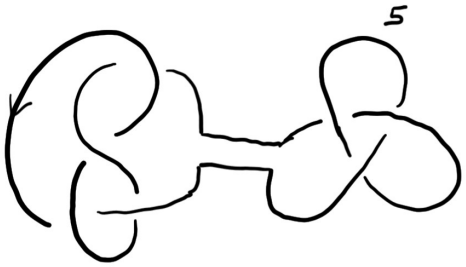
4. Show that the following planar Heegaard diagram represents the 3-manifold $S^1 \times S^2$:



5. Show that surgery along the following two framed links produces diffeomorphic manifolds.



(a)



(b)

6. Explicitly give two Kirby diagrams representing diffeomorphic closed 4-manifolds.