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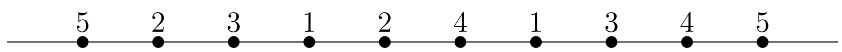
Homework 1 - Topics in Topology

February 28, 2022

Please return before March 17, 2022 at 1pm.

1. Identify the closed, orientable, connected surface determined by the following Kirby diagram:



2. Let $H_g := \natural_g(D^2 \times S^1)$ be the handlebody of genus g, and let $\Sigma_g := \partial H_g$ the genus g surface.

   (a) Show that the union of two solid tori glued along their common boundary $\Sigma_1 = T^2$ with the identity map yields $S^2 \times S^1$. That is, $H_1 \cup_{id} (-H_1) \cong S^2 \times S^1$.

   (b) Draw a planar Heegaard diagram for the previous splitting.

3. Show that the following manifolds are diffeomorphic:

   (a) $L(0,1) \cong S^2 \times S^1$

   (b) $L(2,2021) \cong \mathbb{RP}^3$

   (c) $L(1,2022) \cong S^3$

4. Use the isotopy extension lemma and the argument used in the lectures to show that the (oriented) diffeomorphism type of $M_f := H_g \cup_f (-H_g)$ only depends on the isotopy class of f (you are allowed to omit any consideration about the orientation).

5. Recall that for a topological space X with a choice of basepoint $x_0 \in X$, $\pi_n(X, x_0)$ was defined as the set of basepoint-preserving homotopy classes of basepoint-preserving maps $(S^n, s_0) \to (X, x_0)$, where $s_0 \in S^n$ is some fixed point.

   (a) For a topological space X, show that there is a bijection between $\pi_0(X)$ and the set of path-components of X.

   (b) Write $\mathrm{Diffeo}^+(\Sigma)$ for the set of orientation-preserving diffeomorphisms $\Sigma \to \Sigma$ of an oriented surface Σ. It is possible to endow $\mathrm{Diffeo}^+(\Sigma)$ with a topology (no matter how). Show that there is a bijection between $MCG(\Sigma) \cong \pi_0(\mathrm{Diffeo}^+(\Sigma))$.

6. The goal of this exercise is to prove that the only 3-manifold with a genus 0 Heegaard splitting is S^3 (Theorem 1.4 in Saveliev).

   (a) Prove the Alexander extension lemma: any homeomorphism $S^2 \to S^2$ can be extended to a homeomorphism $D^3 \to D^3$. Use the formula $F(t\mathbf{r}) = tf(\mathbf{r})$ as in the book. Why is it continuous? Why is it a homeomorphism?

(b) Make precise the last sentence of the proof of the theorem: *This homeomorphism can be extended to a homeomorphism from M onto S^3 with the help of Alexander's lemma.*

7. The following exercise is about the notion of "glueing".

   (a) Explain what it means to glue together two topological spaces and be very precise about it (no hand waving!)

   (b) What are the new open sets after gluing?

   (c) Provide a relevant example of your construction.