4.7 The Alexander Grading as a Winding Number

- Topics in Topology -

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Contents

1 Introduction

The goal of this lecture is to relate the Alexander polynomial that we defined in the beginning of this seminar to the grid homologies that were defined previously (in Sections 4.4 and 4.6 in the Grid Homology book).

First by describing the winding number geometrically, we will be able to also geometrically interpret the Alexander grading. This will result in a very practical formula for computing the Alexander grading. Furthermore the Maslov grading will be geometrically interpreted in terms of permutations of grid states.

From this geometric interpretation of the Alexander grading and the Maslov grading we will find a relation between the Alexander polyonomial and grid homology by using the graded Euler characteristic that we will define later on.

2 Prerequisites

Let us first recall some important notions that we will need to use in this lecture.

2.1 Winding numbers

Winding number were defined to construct a knot invariant, they measure the number of windings of a knot around a certain point, taking the orientation into account.

Definition 1. (Definition 3.3.1 in the book.) Let γ be a closed, piecewise linear, oriented curve in $\mathbb{R}^2$ and pick a point $p \in \mathbb{R}^2 \setminus \gamma$. The ***winding number*** $w_\gamma(p)$ of γ around p is defined as the algebraic intersection with the ray ρ from p to the point at infinity.

Based on the winding number the grid matrix was defined.

Definition 2. (Definition 3.3.2 in the book.) For a given grid diagram $\mathbb{G}$, the ***grid matrix*** $M(\mathbb{G})$ is defined by setting

$$(M(\mathbb{G}))_{i,j} = t^{-w((j-1,n-i))}.$$

2.2 The knot invariant $D_{\mathbb{G}}$

By suitably normalizing the determinant of the grid matrix, we found a knot invariant that coincides with the Alexander polynomial.

To this end we defined $a(\mathbb{G})$ to be the sum of the winding numbers of the corners of the $\mathbb{X}$ and $\mathbb{O}$ markings divided by 8 and furthermore $\epsilon(\mathbb{G})$ was defined as the sign of the permutation connecting $\sigma_{\mathbb{O}}$ and $(n, n-1, \dots, 1)$.

Definition 3. (Definition 3.3.4 combined with Lemma's 3.3.7 and 3.3.8 in the book.) The knot invariant $D_{\mathbb{G}}(t)$ is defined as

$$D_{\mathbb{G}}(t) = \epsilon(\mathbb{G}) \cdot \det(M(\mathbb{G})) \cdot (t^{1/2} - t^{-1/2}) t^{a(\mathbb{G})}.$$

Theorem 1. (Theorem 3.3.6 in the book.) *Let $\mathbb{G}$ be a grid diagram for a link L, then $D_{\mathbb{G}}(t)$ is a well-defined link invariant which coincides with the symmetrized Alexander polynomial $\Delta_L(t)$.*

2.3 Bigrading on grid states

The Maslov grading and Alexander grading give rise to a bigrading on the grid states.

Proposition 2. (Proposition 4.3.1 in the book.) *For any toroidal grid diagram $\mathbb{G}$, there is a function*

$$M_{\mathbb{O}} : \mathbf{S}(\mathbb{G}) \to \mathbb{Z},$$

called the **Maslov function on grid states**, *which is uniquely characterized by the following two properties.*

- *Let $\mathbf{x}^{NWO}$ be the grid state whose components are the upper left corners of the squares marked with O. Then*
  $$M_{\mathbb{O}}(\mathbf{x}^{NWO}) = 0.$$
- *If $\mathbf{x}$ and $\mathbf{y}$ are two grid states that can be connected by some rectangle $r \in Rect(\mathbf{x}, \mathbf{y})$, then*
  $$M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{O}}(\mathbf{y}) = 1 - 2\#(r \cap \mathbb{O}) + 2\#(\mathbf{x} \cap Int(r)).$$

Proposition 3. *(Proposition 4.3.7 in the book) Let $\mathbf{x}^{SWO}$ be the grid state whose components are the SW corners of the squares marked with O. Then, $M(\mathbf{x}^{SWO}) = 1 - n$ for any $n \times n$ grid.*

Definition 4. (Definition 4.3.2 in the book.) The ***Alexander function on grid states*** is defined by the formula

$$A(\mathbf{x}) = \frac{1}{2}(M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{X}}(\mathbf{x})) - \Big(\frac{n-1}{2}\Big).$$

2.4 Grid homologies & relations

We will now briefly recall the three different versions of grid homologies that were introduced before.

Definition 5. (Definitions 4.4.1 and 4.4.2 in the book.) The ***fully blocked grid chain complex*** associated to the grid diagram $\mathbb{G}$ is the chain complex $\widetilde{GC}(\mathbb{G})$, with basis over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ given by the grid states $\mathbf{S}(\mathbb{G})$ and differential

$$\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{S}(\mathbb{G})} \#\{r \in \mathrm{Rect}^0(\mathbf{x},\mathbf{y}) | r \cap \mathbb{O} = r \cap \mathbb{X} = \emptyset\} \cdot \mathbf{y}.$$

The ***fully blocked grid homology*** of $\mathbb{G}$, denoted $\widetilde{GH}(\mathbb{G})$, is the corresponding homology of the chain complex $(\widetilde{GH}(\mathbb{G}), \tilde{\partial}_{\mathbb{O},\mathbb{X}})$, thought of as a bigraded vector space.

Definition 6. (Definitions 4.6.1 and 4.6.11 in the book.) The ***(unblocked) grid complex*** $GC^-(\mathbb{G})$ is the free module over $\mathcal{R} = \mathbb{F}[V_1, \ldots, V_n]$ generated by $\mathbf{S}(\mathbb{G})$, with differential

$$\partial^-_{\mathbb{X}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{S}(\mathbb{G})} \sum_{\{r\in\mathrm{Rect}^0(\mathbf{X},\mathbf{y})|r\cap\mathbb{X}=\emptyset\}} V_1^{O_1(r)} \cdots V_n^{O_n(r)} \cdot \mathbf{y}.$$

Fix some $i \in \{1, \ldots, n\}$. The corresponding ***unblocked grid homology*** of $\mathbb{G}$, denoted $GH^-(\mathbb{G})$, is the homology of $(GH^-(\mathbb{G}), \partial^-_{\mathbb{X}})$, viewed as a bigraded module over $\mathbb{F}[\mathbb{U}]$, where the action of U is induced by multiplication by V_i.

Definition 7. (Definition 4.6.12 in the book.) Fix some $i = 1, \ldots n$. The quotient complex $GH^-(\mathbb{G})/V_i$ is called the ***simply blocked grid complex***, and it is denoted $\widehat{GC}(\mathbb{G})$. The ***simply blocked grid homology*** of $\mathbb{G}$, $\widehat{GH}(\mathbb{G})$, is the bigraded vector space obtained as the homology of $\widehat{GH}(\mathbb{G}) = (GH^-(\mathbb{G})/V_i, \partial^-_{\mathbb{X}})$.

The grid homologies that are of interest in this lecture are the fully blocked grid homology of $\mathbb{G}$ and the simply blocked grid homology of $\mathbb{G}$. Section 4.6 in the book provided a way of extracting the vector space $\widehat{GH}(\mathbb{G})$ from the vector space $\widetilde{GH}(\mathbb{G})$.

First let us recall some notation for the tensor product of two bigraded vector spaces. The tensor product of two bigraded vector spaces $X = \bigoplus_{d,s\in\mathbb{Z}} X_{d,x}$ and $Y = \bigoplus_{d,s\in\mathbb{Z}} Y_{d,s}$ is the bigraded vector space

$$(X \otimes Y)_{d,s} = \bigoplus_{d_1+d_2=d, s_1+s_2=s} X_{d_1,s_1} \otimes Y_{d_2,s_2}.$$

Proposition 4. (Proposition 4.6.15 in the book.) *Let $\mathbb{G}$ be a grid diagram representing a knot. Let W be the two-dimensional bigraded vector space, with one generator in bigrading $(0,0)$ and the other in bigrading $(-1,-1)$. Then, there is an isomorphism*

$$\widetilde{GH}(\mathbb{G}) \cong \widehat{GH}(\mathbb{G}) \otimes W^{\otimes(n-1)}$$

of bigraded vector spaces.

3 Geometric interpretation of the winding number

In Section 4.3 of the book, a specific formula for the Maslov grading and Alexander grading of a grid state was proven. This specific formulation of the bigradings made use of the function $\mathcal{I}$ and $\mathcal{J}$ which we will now again need to geometrically interpret the winding number.

The function $\mathcal{I}(P, Q)$ counts the number of pairs $p \in P$ and $q \in Q$ such that $p < q$. We order the points p and q in a grid diagram with respect to the numbering of the vertical and horizontal lines in the grid diagram that was introduced before (see Figure 1 for an example).

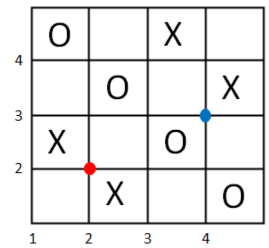

Figure 1: Points p (in red) and q (in blue) such that $p < q$.

The function $\mathcal{J}$ is the symmetric form of the function $\mathcal{I}$, in particular

$$\mathcal{J}(P, Q) = \frac{\mathcal{I}(P, Q) + \mathcal{I}(Q, P)}{2}.$$

The function $\mathcal{J}$ acts on formal linear combinations of (sub)sets by extending it in the following way:

$$\mathcal{J}(P_1 + P_2, Q) := \mathcal{J}(P_1, Q) + \mathcal{J}(P_2, Q).$$

To give some intuition on the functions $\mathcal{I}$ and $\mathcal{J}$, consider the following examples.

Example 1. Consider the simple 2×2 diagram for the unknot as shown in Figure 2. In red the grid state $\mathbf{x}$ is shown. We want to compute $\mathcal{I}(\mathbf{x}, \mathbb{O})$. For the point in the left-bottom corner of the grid state $\mathbf{x}$, there are two $O \in \mathbb{O}$ such that $x < O$ (denotes by the blue lines). For the middle point in the grid state $\mathbf{x}$ there is no $O \in \mathbb{O}$ such that $x < O$. This implies that $\mathcal{I}(\mathbf{x}, \mathbb{O}) = 2$.

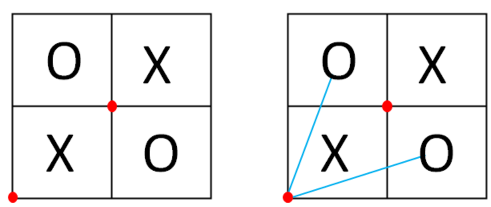

Figure 2: Grid diagram of size 2×2 for the unknot

Example 2. Consider the 3×3 diagram for the unknot as shown in Figure 3. The red dots again denote the grid state $\mathbf{x}$. We want to compute $\mathcal{J}(\mathbf{x}, \mathbb{O})$. The middle picture in Figure 3 shows the pairs (O, x) that contribute to $\mathcal{I}(\mathbb{O}, \mathbf{x})$. The rightmost picture in Figure 3 shows the pairs (x, O) that contribute to $\mathcal{I}(\mathbf{x}, \mathbb{O})$. It follows that

$$\mathcal{J}(\mathbf{x}, \mathbb{O}) = \frac{\mathcal{I}(\mathbf{x}, \mathbb{O}) + \mathcal{I}(\mathbb{O}, \mathbf{x})}{2} = \frac{5 + 2}{2} = 3,5.$$

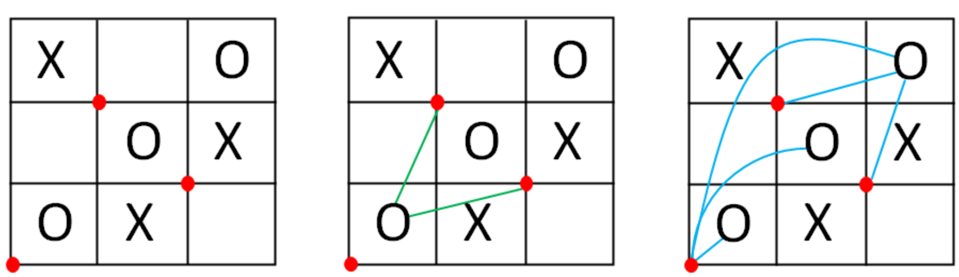

Figure 3: Grid diagram of size 3×3 for the unknot

Lemma 5. *Let $\mathbb{G}$ denote the grid diagram of any knot K and let $\mathcal{D} = \mathcal{D}(\mathbb{G})$ denote the corresponding knot diagram. Furthermore, let p be any point in the diagram not on $\mathcal{D}$. Then*

$$w_{\mathcal{D}}(p) = \mathcal{J}(p, \mathbb{O} - \mathbb{X}). \tag{1}$$

Proof. By definition, the winding number of a point p is defined as counting the intersection points (taking into account the orientation) of a ray ρ (from p to infinity) and the diagram $\mathcal{D}$ of the knot K. We already saw that the winding number of a point p does not depend on the choice of ray ρ.

If $p = (x, y)$ is a point on the grid diagram that is not contained in $\mathcal{D}$ and we consider the ray ρ_+ from p to $(+\infty, y)$, then the winding number of p is determined by the following reasoning. An intersection point of ρ_+ with a vertical line segment of $\mathcal{D}$ pointing upwards (from X to O) contributes $+1$ to the winding number of p. Similarly an intersection point of ρ_+ with a vertical line segment of $\mathcal{D}$ pointing downwards contributes -1 to the winding number of p. A vertical line segment that does not intersect the ray ρ_+ does not contribute anything to the winding number, see Figure 4.

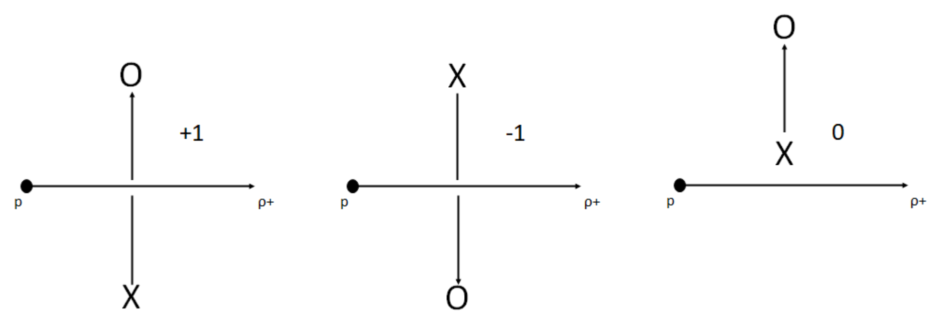

Figure 4: Winding number w.r.t ρ_+

Now we will define notation for four quadrants of the grid diagram depending on the point p, see Figure 5. By using these quadrants we can now say that there is a contribution of $+1$ from a line segment from X to O to the winding number if the O lies in the first quadrant of p and the X does not. In addition, there is a contribution of -1 from a line segment from X to O to the winding number if the X lies in the first quadrant of p and the O does not. For vertical line segments that do not intersect the ray ρ_+, so both X and O lie in the first quadrant or both X and O lie in the second quadrant, there is no contribution to the winding number (see Figure 4).

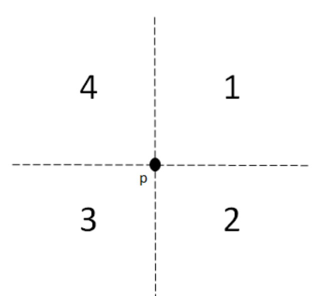

Figure 5: The four quadrants of a point p

This precisely describes $\mathcal{I}(p, \mathbb{O} - \mathbb{X})$, which counts (positive) the number of pairs (p, O) such that O lies in the first quadrant of p and X does not (so X lies in the second quadrant of p), and (negative) the number of pairs (p, X) such that X lies in the first quadrant p and O does not (so O lies in the second quadrant of p), see also Figure 4.

We found that the number of intersection points of the knot with the ray ρ_+ (counted with orientation) equals to $\mathcal{I}(p, \mathbb{O} - \mathbb{X})$, i.e.

$$\#(\rho_+ \cap \mathcal{D}) = \mathcal{I}(p, \mathbb{O} - \mathbb{X}). \tag{2}$$

Similarly, if we take the ray ρ_- from p to (∞, y) instead of ρ_+, we find that the number of intersection points of the knot equals to $\mathcal{I}(\mathbb{O} - \mathbb{X}, p)$,

$$\#(\rho_- \cap \mathcal{D}) = \mathcal{I}(\mathbb{O} - \mathbb{X}, p). \tag{3}$$

Figure 6 show the winding numbers for the ray ρ_+ and the ray ρ_-.

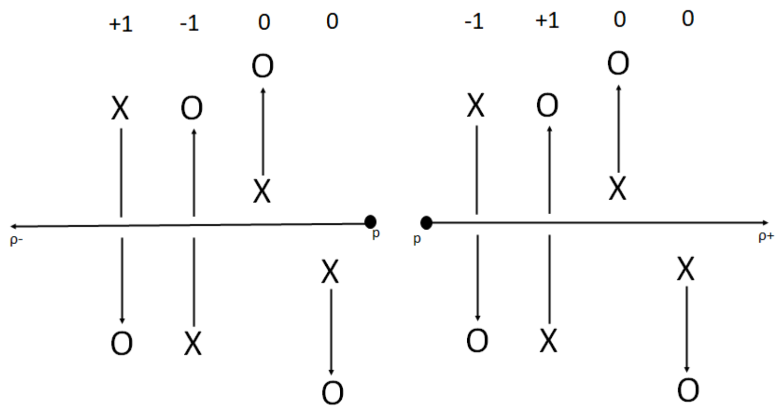

Figure 6: Winding numbers

As was mentioned before, the winding number of a point p does not depend on the choice of ray ρ. This means that $w_{\mathcal{D}}(p) = \#(\rho_+ \cap \mathcal{D}) = \#(\rho_- \cap \mathcal{D})$, and by taking the average of Equations (2) and (3), we find that

$$w_{\mathcal{D}}(p) = \frac{\mathcal{I}(p, \mathbb{O} - \mathbb{X}) + \mathcal{I}(\mathbb{O} - \mathbb{X}, p)}{2} = \mathcal{J}(p, \mathbb{O} - \mathbb{X}).$$

□

4 Geometric interpretation of the Alexander grading

Consider some toroidal grid diagram with a grid state $\mathbf{x}$ for a certain knot K and fix some planar realization $\mathbb{G}$ of the toroidal grid diagram. Let $\mathcal{D}$ denote the knot diagram of the knot K corresponding to the planar grid diagram $\mathbb{G}$.

In order to geometrically interprete the Alexander grading of the grid state $\mathbf{x}$, i.e. express the Alexander grading in terms of the winding numbers, we need to consider a function that sums all the winding numbers of the elements of the grid state. In particular, we define

$$A'(\mathbf{x}) = -\sum_{x \in \mathbf{x}} w_{\mathcal{D}}(x). \tag{4}$$

It turns out that the Alexander grading $A(\mathbf{x})$ and the function $A'(\mathbf{x})$ are closely related. Before stating the exact relation, let us first introduce some notation.

We consider a planar grid diagram $\mathbb{G}$ of size $n \times n$. It has n X markings and n O markings. There are $2n$ squares that contain either an X or an O. Each of these squares has 4 corners, and so the total number of corners of the $2n$ squares equal $8n$ (where possibly some corners are the same). These $8n$ corners correspond to $8n$ points in the grid which we will denote by $p_1, \dots p_{8n}$. Recall from a previous section that we defined a function $a(\mathbb{G})$ as the sum of the winding numbers of the points $p_1, \dots, p_{8n}$.

Proposition 6. *The Alexander function A can be expressed in terms of the winding numbers $w_{\mathcal{D}}$ by means of the following formula*

$$A(\mathbf{x}) = -\sum_{x \in \mathbf{x}} w_{\mathcal{D}}(x) + \frac{1}{8}\sum_{j=1}^{8n} w_{\mathcal{D}}(p_j) - \left(\frac{n-1}{2}\right) = A'(\mathbf{x}) + a(\mathbb{G}) - \frac{n-1}{2}. \tag{5}$$

To prove this Proposition, we will first prove two lemmas.

Lemma 7. *Let $\mathbb{G}$ be an $n \times n$ grid diagram. Consider a square in $\mathbb{G}$ which center z is marked with either an O or an X. Let z_1, z_2, z_3, z_4 denote the four corner points of this square. Then we have*

$$\mathcal{J}(z, \mathbb{O} - \mathbb{X}) = \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, \mathbb{O} - \mathbb{X}) + \begin{cases} \frac{1}{4} & \textit{if } z \textit{ is marked with an } O \\ -\frac{1}{4} & \textit{if } z \textit{ is marked with an } X. \end{cases} \tag{6}$$

Note that there is a mistake in the book, the $\frac{1}{4}$ and $-\frac{1}{4}$ are swapped around.

Proof. First consider the case where the square with center z is marked with O. Then there are $n-1$ remaining O's in the grid. There are also n markings X in the grid, 1 of which is in the same row as the square with center z (we will denote this marking by X_1) and 1 of which is in the same column as the square with center z (we will denote this marking by X_2). We can write

$$\mathcal{J}(z, \mathbb{O} - \mathbb{X}) = \mathcal{J}(z, \mathbb{O}' - \mathbb{X}')$$

Here the set $\mathbb{O}'$ considers all markings in $\mathbb{O}$ that are different from O and the set $\mathbb{X}'$ considers all markings in $\mathbb{X}$ different from X_1 and X_2. For any $O' \in \mathbb{O}'$ it holds that

$$\mathcal{J}(z, O') = \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, O'),$$

and for any $X' \in \mathbb{X}'$, it holds that

$$\mathcal{J}(z, X') = \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, X'),$$

(see Exercise 6). By summing over all $O' \in \mathbb{O}'$ and summing over all $X' \in \mathbb{X}'$, these two equalities imply that

$$\mathcal{J}(z, \mathbb{O}') = \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, \mathbb{O}') \quad \text{and} \quad \mathcal{J}(z, \mathbb{X}') = \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, \mathbb{X}'). \tag{7}$$

However for the markings O, X_1, X_2, this does not hold since $\mathcal{J}(z, O) = \mathcal{J}(z, X_1 + X_2) = 0$, while $\frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, O)$ and $\frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, O)$ certainly do not give the value 0.

We find that

$$\begin{aligned} \mathcal{J}(z, \mathbb{O} - \mathbb{X}) &= \mathcal{J}(z, \mathbb{O}' - \mathbb{X}') \\ &= \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, \mathbb{O}' - \mathbb{X}') \\ &= \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, \mathbb{O} - \mathbb{X}) \\ &\quad - \left[\frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, O) - \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, X_1 + X_2)\right], \end{aligned} \tag{8}$$

i.e., we subtract the 'extra' part coming from the markings O, X_1, X_2 when considering $\frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, \mathbb{O} - \mathbb{X})$.

Now the only thing that remains is computing the terms $\mathcal{J}(z_1 + z_2 + z_3 + z_4, O)$ and $\mathcal{J}(z_1 + z_2 + z_3 + z_4, X_1 + X_2)$. First we will determine $\mathcal{J}(z_1 + z_2 + z_3 + z_4, O)$. We are in the case of Figure 7. From this figure, we see that the only pair that contributes to $\mathcal{I}(z_1 + z_2 + z_3 + z_4, O)$ is the pair (z_4, O) (in red) and the only pair that contributes to $\mathcal{I}(O, z_1 + z_2 + z_3 + z_4)$ is the pair (O, z_2) (in blue). Therefore we have

$$\mathcal{J}(z_1 + z_2 + z_3 + z_4, O) = \frac{\mathcal{I}(z_1 + z_2 + z_3 + z_4, O) + \mathcal{I}(O, z_1 + z_2 + z_3 + z_4)}{2} = \frac{1+1}{2} = 1. \tag{9}$$

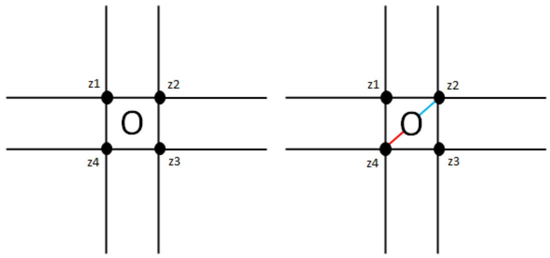

Figure 7: Computing $\mathcal{J}(z_1 + z_2 + z_3 + z_4, O)$

Now we will determine $\mathcal{J}(z_1 + z_2 + z_3 + z_4, X_1 + X_2)$. There are four different cases, which can be seen in Figure 8. The grid diagram always looks like one of those four cases.

- In first case of Figure 8 we see that there are no pairs that contribute to $\mathcal{I}(z_1+z_2+z_3+z_4, X_1+X_2)$ and there are four pairs that contribute to $\mathcal{I}(X_1+X_2, z_1+z_2+z_3+z_4)$, the pairs $(X_1, z_1), (X_1, z_2), (X_2, z_3)$ and (X_2, z_2) (in red).
- In second case of Figure 8 we see that there are two pairs that contribute to $\mathcal{I}(z_1+z_2+z_3+z_4, X_1+X_2)$, the pairs $(z_1, X_2), (z_4, X_2)$ (in blue). There are also two pairs that contribute to $\mathcal{I}(X_1 + X_2, z_1 + z_2 + z_3 + z_4)$, namely the pairs (X_1, z_1) and (X_1, z_2) (in red).
- In third case of Figure 8 we see that there are again two pairs that contribute to $\mathcal{I}(z_1 + z_2 + z_3 + z_4, X_1 + X_2)$, the pairs $(z_3, X_2), (z_4, X_2)$ (in blue). There are also 2 pairs that contribute to $\mathcal{I}(X_1 + X_2, z_1 + z_2 + z_3 + z_4)$, the pairs $(X_1, z_2), (X_1, z_3)$ (in red).
- In last case of Figure 8 we see that there are 4 pairs contributing to $\mathcal{I}(z_1 + z_2 + z_3 + z_4, X_1 + X_2)$, the pairs $(z_1, X_1), (z_4, X_1), (z_4, X_2), (z_3, x_2)$ (in blue) and there are no pairs contributing to $\mathcal{I}(X_1 + X_2, z_1 + z_2 + z_3 + z_4)$.

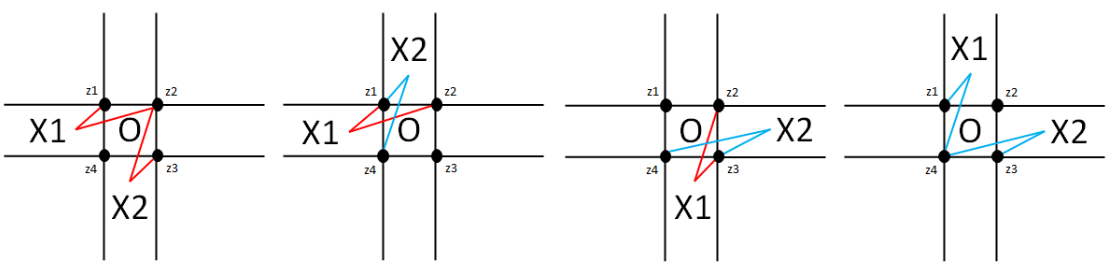

Figure 8: Computing $\mathcal{J}(z_1 + z_2 + z_3 + z_4, X_1 + X_2)$

In all four cases, there are two pairs in total that contribute to either $\mathcal{I}(z_1 + z_2 + z_3 + z_4, X_1 + X_2)$ or $\mathcal{I}(X_1 + X_2, z_1 + z_2 + z_3 + z_4)$. Therefore we find

$$\mathcal{J}(z_1 + z_2 + z_3 + z_4, X_1 + X_2) = \frac{\mathcal{I}(z_1 + z_2 + z_3 + z_4, X_1 + X_2) + \mathcal{I}(X_1 + X_2, z_1 + z_2 + z_3 + z_4)}{2} = \frac{4}{2} = 2. \tag{10}$$

Let us now substitute the results from Equations (7), (9) and (10) into Equation (8). We obtain

$$\begin{aligned}\mathcal{J}(z, \mathbb{O} - \mathbb{X}) &= \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, \mathbb{O} - \mathbb{X}) - \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 2 \\ &= \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, \mathbb{O} - \mathbb{X}) + \frac{1}{4}.\end{aligned}$$

Now consider the case where the square with center z is marked with X. There are $n-1$ remaining X's in the grid and there are n markings O in the grid, 1 of which is in the same row as X and 1 of which is in the same column as X (we will denote these markings by O_1 and O_2). Similar to before, we write

$$\mathcal{J}(z, \mathbb{O}-\mathbb{X}) = \mathcal{J}(z, \mathbb{O}^\star - \mathbb{X}^\star),$$

where the set $\mathbb{O}^\star$ denotes all the markings in $\mathbb{O}$ different from O_1 and O_2 and the set $\mathbb{X}^\star$ denotes all the markings in $\mathbb{X}$ different from X. As for the previous case, we find that

$$\mathcal{J}(z, \mathbb{O}^\star) = \frac{1}{4}\mathcal{J}(z_1+z_2+z_3+z_4, \mathbb{O}^\star) \quad \text{and} \quad \mathcal{J}(z, \mathbb{X}^\star) = \frac{1}{4}\mathcal{J}(z_1+z_2+z_3+z_4, \mathbb{X}^\star), \tag{11}$$

and we can write

$$\begin{aligned}\mathcal{J}(z, \mathbb{O}-\mathbb{X}) = {} & \frac{1}{4}\mathcal{J}(z_1+z_2+z_3+z_4, \mathbb{O}-\mathbb{X}) \\ & - \Big[\frac{1}{4}\mathcal{J}(z_1+z_2+z_3+z_4, O_1+O_2) - \frac{1}{4}\mathcal{J}(z_1+z_2+z_3+z_4, X)\Big].\end{aligned} \tag{12}$$

The computations for $\mathcal{J}(z_1+z_2+z_3+z_4, O_1+O_2)$ and $\mathcal{J}(z_1+z_2+z_3+z_4, X)$ are almost identical to before, so they will not be included here. We find that

$$\mathcal{J}(z_1+z_2+z_3+z_4, O_1+O_2) = 2 \quad \text{and} \quad \mathcal{J}(z_1+z_2+z_3+z_4, X) = 1, \tag{13}$$

so therefore

$$\begin{aligned}\mathcal{J}(z, \mathbb{O}-\mathbb{X}) &= \frac{1}{4}\mathcal{J}(z_1+z_2+z_3+z_4, \mathbb{O}-\mathbb{X}) - \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 1 \\ &= \frac{1}{4}\mathcal{J}(z_1+z_2+z_3+z_4, \mathbb{O}-\mathbb{X}) - \frac{1}{4},\end{aligned}$$

which concludes the proof of Lemma 7. □

Lemma 8. *For a grid diagram $\mathbb{G}$ with corresponding knot diagram $\mathcal{D}$, it holds that*

$$\frac{1}{2}\mathcal{J}(\mathbb{O}+\mathbb{X}, \mathbb{O}-\mathbb{X}) = \frac{1}{8}\sum_{i=1}^{8n} w_{\mathcal{D}}(p_i) \tag{14}$$

Proof. By summing the result from Lemma 2 over all X and O markings in the grid $\mathbb{G}$ we find that

$$\begin{aligned}\mathcal{J}(\mathbb{O}+\mathbb{X}, \mathbb{O}-\mathbb{X}) &= \frac{1}{4}\sum_{i=1}^{8n}\mathcal{J}(p_i, \mathbb{O}-\mathbb{X}) \\ &= \frac{1}{4}\sum_{i=1}^{8n} w_{\mathcal{D}}(p_i),\end{aligned}$$

where we used the result from Lemma 1. From this equation, the result follows directly. □

Now we are finally able to prove Proposition 6.

Proof of Proposition 6. First note that we can sum Equation 1 over all $x \in \mathbf{x}$ to obtain $A'(\mathbf{x}) = -\mathcal{J}(\mathbf{x}, \mathbb{O}-\mathbb{X})$. In T he Alexander grading was defined as

$$A(\mathbf{x}) = \frac{1}{2}\Big(M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{X}}(\mathbf{x})\Big) - \Big(\frac{n-1}{2}\Big). \tag{15}$$

We will now rewrite this expression by using the Maslov grading expressed in terms of the function $\mathcal{J}$.

By substituting this into Equation (15) we obtain

$$
\begin{aligned}
A(\mathbf{x}) &= \frac{1}{2}(\mathcal{J}(\mathbf{x},\mathbf{x}) - 2\mathcal{J}(\mathbf{x},\mathbb{O}) + \mathcal{J}(\mathbb{O},\mathbb{O}) + 1 - \mathcal{J}(\mathbf{x},\mathbf{x}) + 2\mathcal{J}(\mathbf{x},\mathbb{X}) - \mathcal{J}(\mathbb{X},\mathbb{X}) - 1) - \binom{n-1}{2} \\
&= \frac{1}{2}\mathcal{J}(\mathbb{O},\mathbb{O}) - \mathcal{J}(\mathbf{x},\mathbb{O}) + \mathcal{J}(\mathbf{x},\mathbb{X}) - \frac{1}{2}\mathcal{J}(\mathbb{X},\mathbb{X}) - \binom{n-1}{2} \\
&= -\mathcal{J}(x,\mathbb{O}-\mathbb{X}) + \frac{1}{2}(\mathcal{J}(\mathbb{O},\mathbb{O}) - \mathcal{J}(\mathbb{X},\mathbb{X})) - \binom{n-1}{2} \\
&= A'(\mathbf{x}) + \frac{1}{2}\mathcal{J}(\mathbb{O}+\mathbb{X},\mathbb{O}-\mathbb{X}) - \binom{n-1}{2}.
\end{aligned}
$$

Finally, the result of Lemma 8 can be substituted to obtain the desired result

$$
A(\mathbf{x}) = A'(\mathbf{x}) + \frac{1}{8}\sum_{j=1}^{8n} w_{\mathcal{D}}(p_j) - \binom{n-1}{2}. \tag{16}
$$

□

5 Geometric interpretation of the Maslov grading

Proposition 9 gives a relation for the Maslov grading of two different grid states that can be connected by a rectangle. For any two grid states $\mathbf{x}$ and $\mathbf{y}$, there exists a sequence of grid states $\mathbf{x} = \mathbf{x}_1, \ldots, \mathbf{x}_k = y$ and rectangles $r_i \in \mathrm{Rect}(\mathbf{x}_i, \mathbf{x}_{i+1})$. In other words, for any two grid states $\mathbf{x}$ and $\mathbf{y}$ we can always find a sequence of grid states from $\mathbf{x}$ to $\mathbf{y}$ that can be connected by rectangles. The length of this sequence of rectangles determines the number of transpositions in the permutation from grid state $\mathbf{x}$ to grid state $\mathbf{y}$. This is precisely the idea behind Lemma 9.

Example 3. We consider the 4×4 diagram of the Borromean rings as depicted in Figure 9. Let $\mathbf{x}$ be the grid state in red, $\mathbf{y}$ be the grid state in green and $\mathbf{z}$ be the grid state in purple. First note that the grid states $\mathbf{x}$ and $\mathbf{z}$ can not be connected by a rectangle since they differ in all points. Grid states $\mathbf{x}$ and $\mathbf{y}$ differ in only 2 points, they can be connected by a rectangle (for example as in the middle picture in Figure 9). Grid states $\mathbf{y}$ and $\mathbf{z}$ also differ in only 2 points, so they be connected by a rectangle (for example as in the rightmost picture in Figure 9). This means that we now have a sequence of grid states $\mathbf{x}, \mathbf{y}, \mathbf{z}$ that can be connected by a rectangle. Note that since the sequence has length 3, the permutation from $\mathbf{x}$ to $\mathbf{z}$ consists of 2 transpositions.

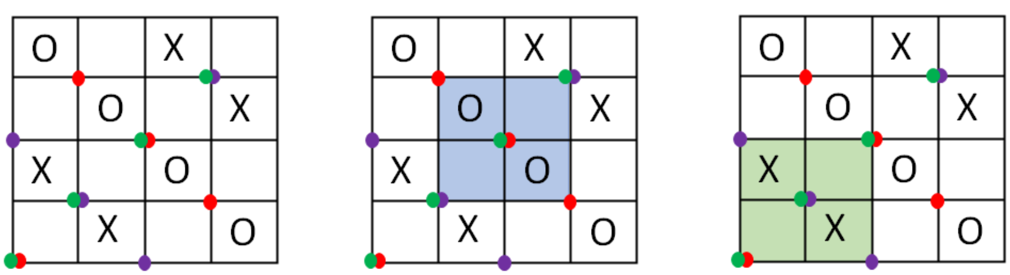

Figure 9: Rectangles connecting grid states

Lemma 9. (Lemma 4.7.3 in the book.) *The sign of the permutation that connects* $\mathbf{x}$ *with* $\mathbf{x}^{NWO}$ *is* $(-1)^{M(\mathbf{x})}$.

Proof. First recall that the sign of a permutation is defined as $(-1)^m$, where m is the number of transpositions in the decomposition into transpositions of the permutation.

Suppose that there is a sequence of length k of grid states connecting grid state $\mathbf{x}$ with grid state $\mathbf{x}^{NWO}$, i.e. we have

$$\mathbf{x} = \mathbf{x}_1, \ldots, \mathbf{x}_k = \mathbf{x}^{NWO}.$$

Then the number $k-1$ is equal to the number of transposition in the permutations from $\mathbf{x}$ to $\mathbf{y}$, and so the sign of the permutation that connects $\mathbf{x}$ with $\mathbf{x}^{NWO}$ is equal to $(-1)^{k-1}$.

For the subsequent pairs of grid states in the sequence of grid states from $\mathbf{x}$ to $\mathbf{x}^{NWO}$ we find that

$$\begin{aligned} M(\mathbf{x}) - M(\mathbf{x}_2) &\equiv 1 \mod 2 \\ M(\mathbf{x}_2) - M(\mathbf{x}_3) &\equiv 1 \mod 2 \\ &\vdots \\ M(\mathbf{x}_{k-1}) - M(\mathbf{x}^{NWO}) &\equiv 1 \mod 2, \end{aligned}$$

using Equation (4.2) in Proposition 4.3.1 from the book. This is a system of $k-1$ equations. By adding all the equations and using that $M(\mathbf{x}^{NWO}) = 0$ (from Equation (4.1) in Proposition 4.3.1), we find that

$$M(\mathbf{x}) \equiv k-1 \mod 2.$$

This implies that the number of transposition of the permutation that connects $\mathbf{x}$ with $\mathbf{x}^{NWO}$ is equal to $M(\mathbf{x})$. This indeed implies that the sign of the permutation that connects $\mathbf{x}$ with $\mathbf{x}^{NWO}$ is $(-1)^{M(\mathbf{x})}$.

Note that we can take the mod 2 reduction of Equation (4.2) in Proposition 9 since we only care about the sign. □

6 Euler characteristic

In order to relate the Alexander polynomial of a knot to its grid homology we will now define the graded Euler characteristic of a bigraded vector space.

Definition 8. (Definition 4.7.4 in the book.) Let $X = \bigoplus_{d,s} X_{d,s}$ be a bigraded vector space. We define the ***graded Euler characteristic*** of X to be the Laurent polynomial in t given by

$$\chi(X) = \sum_{d,s} (-1)^d \dim X_{d,s} \cdot t^s$$

6.1 Euler characteristic for the fully blocked grid homology

Now the Euler characteristic of the fully blocked grid homology can be related to the Alexander polynomial.

Proposition 10. (Proposition 4.7.5 in the book.) *Let $\mathbb{G}$ be an $n \times n$ grid diagram for a knot K. The graded Euler characteristic of the bigraded vector space $\widetilde{GH}(\mathbb{G})$ is given by*

$$\chi(\widetilde{GH}(\mathbb{G})) = (1-t^{-1})^{n-1} \cdot \Delta_K(t),$$

where $\Delta_K(t)$ denotes the symmetrized Alexander polynomial of the knot K.

In the proof of this proposition we will make use of two lemmas.

Lemma 11. *For a variable t and an integer n, it holds that*

$$t^{\frac{1-n}{2}}(-1)^{n-1} = (1-t^{-1})^{n-1}(t^{-1/2} - t^{1/2})^{1-n}.$$

Proof. Write out the right-hand side of the equation

$$\begin{aligned}(1-t^{-1})^{n-1}(t^{-1/2}-t^{1/2})^{1-n} &= t^{\frac{n-1}{2}}t^{\frac{1-n}{2}}(1-t^{-1})^{n-1}(t^{-1/2}-t^{1/2})^{1-n} \\ &= t^{-\frac{1}{2}(1-n)}r^{\frac{1-n}{2}}(1-t^{-1})^{n-1}(t^{-1/2}-t^{1/2})^{1-n} \\ &= t^{\frac{1-n}{2}}(1-t^{-1})^{n-1}(t^{-1/2}(t^{-1/2}-t^{1/2}))^{1-n} \\ &= t^{\frac{n-1}{2}}(1-t^{-1})^{n-1}(t^{-1}-1)^{1-n} \\ &= t^{\frac{n-1}{2}}(-1)^{n-1}(t^{-1}-1)^{n-1}(t^{-1}-1)^{1-n} \\ &= t^{\frac{1-n}{2}}(-1)^{n-1},\end{aligned}$$

which yields the desired equality. □

Lemma 12. *Let $\mathbb{G}$ be an $n \times n$ grid diagram. Then it holds that*

$$\sum_{\mathbf{x}\in S(\mathbb{G})}(-1)^{M(\mathbf{x})}t^{-\sum_{x\in\mathbf{x}}w_{\mathcal{D}}(x)} = (-1)^{n-1}\epsilon(\mathbb{G})\det(M(\mathbb{G})).$$

Before proving this lemma, recall that the determinant of an $n \times n$ matrix can be written in terms of permutations. Suppose that A is an $n \times n$ matrix. Then we have

$$\det(A) = \sum_{\sigma\in S_n}\operatorname{sign}(\sigma)a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}.$$

This means that the determinant of the grid matrix can be viewed as a weighted count of permutations, the weight is obtained as a monomial in t, with exponent given by a winding number.

Proof. From Lemma 9 we know that $(-1)^{M(\mathbf{x})}$ is equal to the sign of the permutation that connects $\mathbf{x}$ with $\mathbf{x}^{NWO}$. This means that we can write

$$\begin{aligned}\sum_{\mathbf{x}\in S(\mathbb{G})}(-1)^{M(\mathbf{x})}t^{-\sum_{x\in\mathbf{x}}w_{\mathcal{D}}(x)} &= \sum_{\mathbf{x}\in S(\mathbb{G})}\operatorname{sign}(\mathbf{x}^{NWO})\operatorname{sign}(\mathbf{x})t^{-\sum_{x\in\mathbf{x}}w_{\mathcal{D}}(x)} \\ &= \operatorname{sign}(\mathbf{x}^{NWO})\sum_{\mathbf{x}\in S(\mathbb{G})}\operatorname{sign}(\mathbf{x})t^{-\sum_{x\in\mathbf{x}}w_{\mathcal{D}}(x)}.\end{aligned}$$

Now the sum is almost equal to the determinant of the grid matrix. However, we need to take into account the correspondence between entries in the grid matrix and lattice point in the grid diagram. The entry (i,j) in the grid matrix corresponds to the lattice point $(j-1, n-i)$ in the grid diagram. The identity permutation of the matrix corresponds to the diagonal, while the corresponding identity permutation of the grid state corresponds to the anti-diagonal. These two permutations are related via the permutation $(n, n-1, \dots, 1)$. So in order to obtain the determinant of the grid matrix we should include the sign of $(n, n-1, \dots, 1)$ and the sign of its inverse (which is equal to itself) into the expression above. We obtain

$$\begin{aligned}\sum_{\mathbf{x}\in S(\mathbb{G})}(-1)^{M(\mathbf{x})}t^{-\sum_{x\in\mathbf{x}}w_{\mathcal{D}}(x)} &= \operatorname{sign}(\mathbf{x}^{NWO})\operatorname{sign}(n, n-1\dots,1) \\ &\quad\cdot\sum_{\mathbf{x}\in S(\mathbb{G})}\operatorname{sign}(\mathbf{x})\operatorname{sign}(n,n-1\dots,1)t^{-\sum_{x\in\mathbf{x}}w_{\mathcal{D}}(x)} \\ &= \operatorname{sign}(\mathbf{x}^{NWO})\operatorname{sign}(n,n-1\dots,1)\det(M(\mathbb{G})).\end{aligned} \tag{17}$$

The grid state $\mathbf{x}^{NWO}$ is related to the grid state $\mathbf{x}^{SWO}$ via a permutation τ, i.e. we have $\mathbf{x}^{SWO} = \tau\mathbf{x}^{NWO}$. Then by taking the signs we obtain $\operatorname{sign}(\mathbf{x}^{SWO}) = \operatorname{sign}(\tau)\operatorname{sign}(\mathbf{x}^{NWO})$. By Lemma 9 the sign of τ is equal to $(-1)^{M(\mathbf{x}^{SWO})}$ and from Proposition 3 $M(\mathbf{x}^{SWO}) = 1-n$. This yields

$$\operatorname{sign}(\mathbf{x}^{SWO}) = (-1)^{1-n}\operatorname{sign}(\mathbf{x}^{NWO}),$$

or equivalently

$$\text{sign}(\mathbf{x}^{NWO}) = (-1)^{n-1}\text{sign}(\mathbf{x}^{SWO}).$$

By substituting this result into Equation (17), we find that

$$\sum_{\mathbf{x}\in S(\mathbb{G})} (-1)^{M(\mathbf{x})} t^{-\sum_{x\in\mathbf{x}} w_{\mathcal{D}}(x)} = (-1)^{n-1}\text{sign}(\mathbf{x}^{SWO})\text{sign}(n, n-1\ldots,1)\det(M(\mathbb{G})).$$

Now note that the permutation corresponding to grid state $\mathbf{x}^{SWO}$ is exactly equal to the permutation of $\mathbb{O}$. This means that the $\text{sign}(\mathbf{x}^{SWO})\text{sign}(n, n-1\ldots,1)$ is the sign of the permutation that connects $\sigma_{\mathbb{O}}$ and $(n, n-1, \ldots, 1)$, which is equal to $\epsilon(\mathbb{G})$. Therefore

$$\sum_{\mathbf{x}\in S(\mathbb{G})} (-1)^{M(\mathbf{x})} t^{-\sum_{x\in\mathbf{x}} w_{\mathcal{D}}(x)} = (-1)^{n-1}\epsilon(\mathbb{G})\det(M(\mathbb{G})).$$

□

Proof of Proposition 10. Using the definition of the Euler characteristic of a bigraded vector space, if follows that

$$\chi(\widetilde{GH}(\mathbb{G})) = \sum_{d,s\in\mathbb{Z}} (-1)^d \dim\widetilde{GH}_d(\mathbb{G}, s)\cdot t^s.$$

The Euler characteristic of a chain complex is equal to the Euler characteristic of its homology (see Exercise 3), which yields

$$\chi(\widetilde{GH}(\mathbb{G})) = \sum_{d,s\in\mathbb{Z}} (-1)^d \dim\widetilde{GC}_d(\mathbb{G}, s)\cdot t^s.$$

In stead of summing over the pairs of integers $d, s \in \mathbb{Z}$ we can also sum over the corresponding grid state. Then each chain complex is generated by only 1 grid state and we thus find that

$$\chi(\widetilde{GH}(\mathbb{G})) = \sum_{\mathbf{x}\in S(\mathbb{G})} (-1)^{M(\mathbf{x})} t^{A(\mathbf{x})}.$$

Proposition 6 and rewriting the expression gives

$$\begin{aligned}\chi(\widetilde{GH}(\mathbb{G})) &= \sum_{\mathbf{x}\in S(\mathbb{G})} (-1)^{M(\mathbf{x})} t^{A'(\mathbf{x})+a(\mathbb{G})-\left(\frac{n-1}{2}\right)} \\ &= \sum_{\mathbf{x}\in S(\mathbb{G})} (-1)^{M(\mathbf{x})} t^{a(\mathbb{G})} t^{\left(\frac{n-1}{2}\right)} t^{A'(\mathbf{x})} \\ &= t^{a(\mathbb{G})} t^{\left(\frac{n-1}{2}\right)} \sum_{\mathbf{x}\in S(\mathbb{G})} (-1)^{M(\mathbf{x})} t^{-\sum_{x\in\mathbf{x}} w_{\mathcal{D}}(x)}.\end{aligned}$$

By using the result from Lemma 12, if follows that

$$\chi(\widetilde{GH}(\mathbb{G})) = t^{a(\mathbb{G})} t^{\left(\frac{n-1}{2}\right)} (-1)^{n-1}\det(M(\mathbb{G}))\epsilon(\mathbb{G}).$$

Then using the result from Lemma 11 yields

$$\chi(\widetilde{GH}(\mathbb{G})) = (1-t^{-1})^{n-1}(t^{-1/2}-t^{1/2})^{1-n} t^{a(\mathbb{G})}\det(M(\mathbb{G}))\epsilon(\mathbb{G}).$$

From this equation, we recognize the knot invariant $D_{\mathbb{G}}(t)$, and using that $D_{\mathbb{G}}(t) = \Delta_K(t)$ gives the final result

$$\chi(\widetilde{GH}(\mathbb{G})) = (1-t^{-1})^{n-1}\cdot\Delta_K(t).$$

□

6.2 Euler characteristic for the simply blocked grid homology

Proposition 4 provides an isomorphism relation between the fully blocked grid homology and the simply blocked grid homology of a grid diagram. This proposition will be essential in the relation between the Euler characteristic for the simply blocked grid homology and the Alexander polynomial.

Theorem 13. (Theorem 4.7.6 in the book.) *The graded Euler characteristic of the simply blocked grid homology is equal to the (symmetrized) Alexander polynomial $\Delta_K(t)$:*

$$\chi(\widehat{GH}(\mathbb{G})) = \Delta_K(t).$$

Proof. Proposition 4 provides a relation between the Euler characteristics of bigraded vector spaces (see Exercise 5):

$$\chi(\widetilde{GH}(\mathbb{G})) = \chi(\widehat{GH}(\mathbb{G})) \cdot \chi(W)^{n-1}. \tag{18}$$

The vector space W is a 2-dimensional bigraded vector space with one generator in bigrading $(0,0)$ and the other generator in bigrading $(-1,-1)$. Then the Euler characteristic of W can be computed as

$$\chi(W) = \sum_{d,s}(-1)^d \dim W_{d,s} t^s = 1 - t^{-1}.$$

Proposition 10 provides the graded Euler characteristic of the fully blocked grid homology

$$\chi(\widetilde{GH}(\mathbb{G})) = (1 - t^{-1})^{n-1} \cdot \Delta_K(t).$$

Substituting the graded Euler characteristics of W and $\widetilde{GH}(\mathbb{G})$ into Equation 18 yields

$$(1 - t^{-1})^{n-1} \cdot \Delta_K(t) = \chi(\widehat{GH}(\mathbb{G})) \cdot (1 - t^{-1})^{n-1},$$

which implies that

$$\chi(\widehat{GH}(\mathbb{G})) = \Delta_K(t).$$

□

7 Exercises

Exercise 1. Consider the grid diagram of size 5×5 for the trefoil knot in Figure 10 with grid state $\mathbf{x}$ denoted in red. For each $x \in \mathbf{x}$, verify that $w_{\mathcal{D}}(x) = \mathcal{J}(x, \mathbb{O} - \mathbb{X})$.

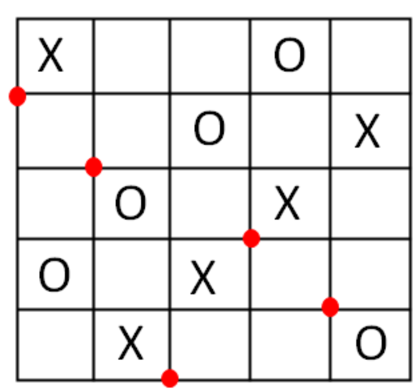

Figure 10: Grid diagram for trefoil knot

Exercise 2. Compute the Alexander grading of the grid state $\mathbf{x}$ in Figure 10 with by means of the formula from Proposition 6.

Exercise 3. Show that the (graded) Euler characteristic of a chain complex is equal to the (graded) Euler characteristic of its homology.

Exercise 4. By computing the Alexander polynomial of the trefoil and the Euler characteristic of the simply blocked grid homology of the trefoil, verify that they are equal (see Figure 10 for the grid diagram of the trefoil knot).

Exercise 5. Let $X = \bigoplus_{d,s} X_{d,s}$ and $Y = \bigoplus_{d,s} Y_{d,s}$ be two bigraded vector spaces. Prove that

$$\chi(X \otimes Y) = \chi(X) \cdot \chi(Y).$$

Exercise 6. (Part of the proof of Lemma 6.) Let $\mathbb{G}$ denote a grid diagram and consider a square with center z that is marked with O. There is one X marking in the same row as the square with center z, call this marking X_1 and there is one X marking in the same column as the square with center z, call this marking X_2. Let $\mathbb{O}'$ denote the set of all markings in $\mathbb{O}$ different from O. Let $\mathbb{X}'$ denote all the markings in $\mathbb{X}$ different from X_1 and X_2. By considering all different cases, prove the following statements. For any $O' \in \mathbb{O}'$, it holds that

$$\mathcal{J}(z, O') = \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, O').$$

(The case for $X' \in \mathbb{X}$ is exactly the same.)

8 References

[1] Peter S Ozsváth, András I Stipsicz, and Zoltán Szabó. *Grid homology for knots and links*, volume 208. American Mathematical Soc., 2017.