The Alexander Grading as a Winding Number

Topics in Topology

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Introduction

Theorem (4.7.6)

The graded Euler characteristic of the simply blocked grid homology is equal to the (symmetrized) Alexander polynomial Δ_K:

$$\chi(\widehat{GH}(\mathbb{G})) = \Delta_K(t).$$

Overview

- Some prerequisites
- Geometric interpretation of the winding number
- Geometric interpretation of the Alexander grading
- Geometric interpretation of the Maslov grading
- The graded Euler characteristic
- Exercises

Definition - Winding numbers (3.3.1)

Let γ be a closed, piecewise linear, oriented curve in $\mathbb{R}^2$ and pick a point $p \in \mathbb{R}^2 \setminus \gamma$. The **winding number** $w_\gamma(p)$ of γ around p is defined as the algebraic intersection with the ray ρ from p to the point at infinity.

- independent of choice of ray

Prerequisites

- Grid matrix $M(\mathbb{G})$ $\quad \epsilon^{(-1)\,\mathrm{WD}(x)}$
- $a(\mathbb{G})$ $= 1/8 \sum \mathrm{WD}(p^q)$
- $\epsilon(\mathbb{G})$ $\quad \mathrm{sign}(\mathbb{O}\mathbb{O} \to (n, n-1, .. 1))$

Definition - $D_{\mathbb{G}}(t)$ (3.3.4)

The knot invariant $D_{\mathbb{G}}(t)$ is defined as

$$D_{\mathbb{G}}(t) = \epsilon(\mathbb{G}) \cdot \det(M(\mathbb{G})) \cdot (t^{1/2} - t^{-1/2})^{1-n} t^{a(\mathbb{G})}.$$

Theorem (3.3.6)

Let $\mathbb{G}$ be a grid diagram for a knot K, then $D_{\mathbb{G}}(t)$ coincides with the symmetrized Alexander polynomial $\Delta_K(t)$.

Prerequisites

$$m(x^{NWO}) = 0$$

- Maslov grading
  $$M(x) - M(y) = 1 - 2\#(r \cap \mathbb{O}) + 2\#(\mathrm{Int}(r) \cap x)$$
  $$M(x) = \mathcal{J}(x,x) - 2\mathcal{J}(x,\mathbb{O}) + \mathcal{J}(\mathbb{O},\mathbb{O}) + 1$$
- Alexander grading $A(x) = \frac{1}{2}(M_{\mathbb{O}}(x) - M_{\mathbb{X}}(x)) - \left(\frac{n-1}{2}\right)$
- Fully blocked grid homology of $\mathbb{G}$: $\widetilde{GH}(\mathbb{G})$
- Simply blocked grid homology of $\mathbb{G}$: $\widehat{GH}(\mathbb{G})$

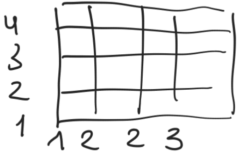
Proposition (4.6.15)

Let $\mathbb{G}$ be a grid diagram representing a knot. Let W be the two-dimensional bigraded vector space, with one generator in bigrading $(0, 0)$ and the other in bigrading $(-1, -1)$. Then, there is an isomorphism

$$\widetilde{GH}(\mathbb{G}) \cong \widehat{GH}(\mathbb{G}) \otimes W^{\otimes(n-1)}.$$

of bigraded vector spaces.

Geometric Interpretation of the Winding Number



- The function $\mathcal{I}(P, Q)$ counts the pairs of points $p \in P$ and $q \in Q$ such that $p < q$.
- Ordering of points in a grid diagram
- The function $\mathcal{J}$ is the symmetric form of $\mathcal{I}$

$$\mathcal{J}(P, Q) = \frac{\mathcal{I}(P, Q) + \mathcal{I}(Q, P)}{2}.$$

The functions $\mathcal{I}$ and $\mathcal{J}$

Example

Compute $\mathcal{J}(\mathbf{x}, \mathbb{O})$.

$$= \frac{\mathcal{I}(x,\mathbb{O}) + \mathcal{I}(\mathbb{O},x)}{2} = \frac{5 + 2}{2} = 3.5$$

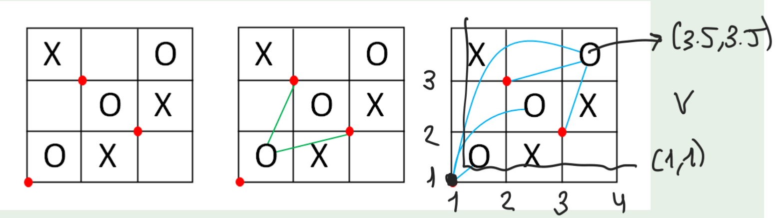


Figure: Grid diagram of size 3×3 for the unknot

Winding number in terms of $\mathcal{J}$

Lemma (4.7.1)

Let $\mathbb{G}$ denote the grid diagram of any knot K and let $\mathcal{D} = \mathcal{D}(\mathbb{G})$ denote the corresponding knot diagram. Furthermore, let p be any point in the diagram not on $\mathcal{D}$. Then

$$w_{\mathcal{D}}(p) = \mathcal{J}(p, \mathbb{O} - \mathbb{X}).$$

Winding number in terms of $\mathcal{J}$

Lemma (4.7.1)

Let $\mathbb{G}$ denote the grid diagram of any knot K and let $\mathcal{D} = \mathcal{D}(\mathbb{G})$ denote the corresponding knot diagram. Furthermore, let p be any point in the diagram not on $\mathcal{D}$. Then

$$w_{\mathcal{D}}(p) = \mathcal{J}(p, \mathbb{O} - \mathbb{X}).$$

Proof: See hand-out / book. $= \mathcal{J}(p, \mathbb{O}) - \mathcal{J}(p, \mathbb{X})$

Geometric Interpretation of the Alexander Grading

- Define $A'(\mathbf{x}) = -\sum_{x \in \mathbf{x}} w_{\mathcal{D}}(x)$.
- Notation: the $8n$ corners of squares with a marking are denoted by $p_1, \ldots, p_{8n}$.

Proposition (4.7.2)

The Alexander function can be expressed in terms of the winding numbers $w_{\mathcal{D}}$ by means of the following formula

$$A(\mathbf{x}) = -\sum_{x \in \mathbf{x}} w_{\mathcal{D}}(x) + \frac{1}{8} \sum_{j=1}^{8n} w_{\mathcal{D}}(p_j) - \left(\frac{n-1}{2}\right)$$

$$= A'(\mathbf{x}) + a(\mathbb{G}) - \left(\frac{n-1}{2}\right).$$

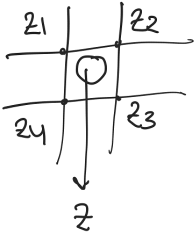
Lemma

Let $\mathbb{G}$ be an $n \times n$ grid diagram. Consider a square in $\mathbb{G}$ which center z is marked with either an O or an X. Let z_1, z_2, z_3, z_4 denote the four corner points of this square. Then we have

$$\mathcal{J}(z, \mathbb{O}-\mathbb{X}) = \frac{1}{4}\mathcal{J}(z_1+z_2+z_3+z_4, \mathbb{O}-\mathbb{X}) + \begin{cases} \frac{1}{4} & \text{if } z \text{ marked with } O \\ -\frac{1}{4} & \text{if } z \text{ marked with } X. \end{cases}$$

↳ mistake in the book

Proof: $\mathbb{G}$



$n-1$ remaining O's $\quad \mathbb{O}'$

n X's markings

- X_1, X_2
- $n-2$ remaining X's $\quad \mathbb{X}'$

$$g(z, \mathbb{O} - \mathbb{X}) = g(z, \mathbb{O}' - \mathbb{X}') \leftarrow$$

$O' \in \mathbb{O}'$ $\quad g(z, O') = \frac{1}{4} g(z_1 + \cdots z_4, O')$

$X' \in \mathbb{X}'$ $\quad g(z, X') = \frac{1}{4} g(z_1 + \cdots z_4, X')$

(exercise)

Proof (cont.):

$$g(z, \mathbb{O}' - \mathbb{X}') = \tfrac{1}{4} g(z_1 + \cdots z_4, \mathbb{O}' - \mathbb{X}')$$

$$g(z, \mathbb{O} - \mathbb{X}) = \tfrac{1}{4} g(z_1 + \cdots + z_4, \underline{\underline{\mathbb{O}' - \mathbb{X}'}})$$

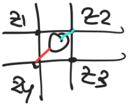
$$= \underbrace{\tfrac{1}{4} g(z_1 + \cdots z_4, \mathbb{O} - \mathbb{X})}_{\text{too large}}$$

$$- \tfrac{1}{4} \Big[\underbrace{g(z_1 + \cdots z_4, O)}_{} - \underbrace{g(z_1 + \cdots + z_4, X_1 + X_2)}_{} \Big]$$

$$= \tfrac{1}{4} g(z_1 + \cdots z_4, \mathbb{O} - \mathbb{X}) + 1/4 \qquad \square$$

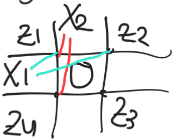
Proof (cont.):

> $\mathcal{A}(z_1+\cdots z_4, O) = \dfrac{I(z_1+\cdots z_4, O) + I(O, z_1\cdots z_4)}{2}$



$= \dfrac{1+1}{2} = 1$

> $\mathcal{A}(z_1+\cdots z_4, X_1+X_2) = \dfrac{I(z_1+\cdots z_4, X_1+X_2) + I(X_1+X_2, \cdots)}{2}$



$= \dfrac{2+2}{2}$

$= 2$

Proof (cont.):

Lemma

For a grid diagram $\mathbb{G}$ with corresponding knot diagram $\mathcal{D}$, it holds that

$$\frac{1}{2}\mathcal{J}(\mathbb{O}+\mathbb{X}, \mathbb{O}-\mathbb{X}) = \frac{1}{8}\sum_{i=1}^{8n} w_{\mathcal{D}}(p_i).$$

Proof. $\mathcal{J}(z, \mathbb{O}-\mathbb{X}) = \frac{1}{4}\mathcal{J}(z_1+\cdots+z_4, \mathbb{O}-\mathbb{X}) \underline{\underline{\pm 1/4}}$

$$\mathcal{J}(\mathbb{O}+\mathbb{X}, \mathbb{O}-\mathbb{X}) = \frac{1}{4}\sum_{i=1}^{8n} \underbrace{\mathcal{J}(p_i, \mathbb{O}-\mathbb{X})}_{w_{\mathcal{D}}(p_i)}$$

$$= \frac{1}{4}\sum_{i=1}^{8n} w_{\mathcal{D}}(p_i) \qquad \square$$

$(x-a)(x+a)$

x^2-a^2

Proof (Proposition): Want to show:

$$A(x) = A'(\mathbf{x}) + a(\mathbb{G}) - \left(\frac{n-1}{2}\right).$$

$$A(x) = \frac{1}{2}\left(M_{O}(x) - M_{X}(x)\right) - \left(\frac{n-1}{2}\right)$$

$$= \frac{1}{2}\Big(\cancel{\mathcal{J}(x,x)} - 2\mathcal{J}(x,\mathbb{O}) + \mathcal{J}(\mathbb{O},\mathbb{O}) \cancel{+1}$$

$$- \cancel{\mathcal{J}(x,x)} + 2\mathcal{J}(x,\mathbb{X}) - \mathcal{J}(\mathbb{X},\mathbb{X}) \cancel{-1}\Big) - \left(\frac{n-1}{2}\right)$$

$$= -\mathcal{J}(x,\mathbb{O}) + \mathcal{J}(x,\mathbb{X}) + \frac{1}{2}\left(\mathcal{J}(\mathbb{O},\mathbb{O}) - \mathcal{J}(\mathbb{X},\mathbb{X})\right) - \left(\frac{n-1}{2}\right)$$

$$= -\underbrace{\mathcal{J}(x,\mathbb{O}-\mathbb{X})} + \frac{1}{2}\underbrace{\left(\mathcal{J}(\mathbb{O}+\mathbb{X},\mathbb{O}-\mathbb{X})\right)} - \left(\frac{n-1}{2}\right) \qquad (*)$$

$$A'(x) = -\sum WD(x)$$
$$= -\sum g(x, \mathbb{O} - x)$$
→ point in grid state
$$= -g(x, \mathbb{O} - \mathbf{x})$$
↳ bold x, grid state

$$\overset{(*)}{=} A'(x) + \underbrace{\frac{1}{8}\sum WD(p^{f})} - \left(\frac{n-1}{2}\right)$$

$$= A'(x) + a(\mathbb{O}) - \left(\frac{n-1}{2}\right) \qquad \square$$

Geometric Interpretation of the Maslov Grading

Maslov grading and permutations

$x = x_1, \ldots x_n = y$

- Sequences of grid states connected by rectangles
- Permutations

Lemma

The sign of the permutation that connects $\mathbf{x}$ with $\mathbf{x}^{NWO}$ is $(-1)^{M(\mathbf{x})}$.

- Sequences of grid states connected by rectangles
- Permutations

Lemma

The sign of the permutation that connects $\mathbf{x}$ with $\mathbf{x}^{NWO}$ is $(-1)^{M(\mathbf{x})}$.

Proof: See hand-out/book

Euler characteristic

Definition - Graded Euler characteristic (4.7.4)

Let $X = \bigoplus_{d,s} X_{d,s}$ be a bigraded vector space. We define the **graded Euler characteristic** of X to be the Laurent polynomial in t given by

$$\chi(X) = \sum_{d,s} (-1)^d \dim X_{d,s} \cdot t^s.$$

Proposition (4.7.5)

Let $\mathbb{G}$ be an $n \times n$ grid diagram for a knot K. The graded Euler characteristic of the bigraded vector space $\widetilde{GH}(\mathbb{G})$ is given by

$$\chi(\widetilde{GH}(\mathbb{G})) = (1 - t^{-1})^{n-1} \cdot \Delta_K(t).$$

Euler characteristic for $\widetilde{GH}(\mathbb{G})$

Lemma

For a variable t and an integer n, it holds that

$$t^{\frac{1-n}{2}}(-1)^{n-1} = (1-t^{-1})^{n-1}(t^{-1/2}-t^{1/2})^{1-n}.$$

Lemma

Let $\mathbb{G}$ be an $n \times n$ grid diagram. Then it holds that

$$\sum_{\mathbf{x}\in S(\mathbb{G})} (-1)^{M(\mathbf{x})} t^{-\sum_{x\in\mathbf{x}} w_{\mathcal{D}}(x)} = (-1)^{n-1}\epsilon(\mathbb{G})\det(M(\mathbb{G})).$$

Proof (Proposition): Want to show

$$\chi(\widetilde{GH}(\mathbb{G})) = (1 - t^{-1})^{n-1} \cdot \Delta_K(t).$$

$$\chi(\widetilde{GH}(\mathbb{G})) = \sum_{d,s} (-1)^d \dim \underline{\widetilde{GH}}_d(\mathbb{G}, s)\, t^s$$

exercise

$$= \sum_{d,s} (-1)^d \dim \widetilde{GC}_d(\mathbb{G}, s)\, t^s$$

$$= \sum_{x} (-1)^{M(x)}\, t^{A(x)}$$

$$= \sum_{x} (-1)^{M(x)}\, t^{A'(x) + a(\mathbb{G}) - \left(\frac{n-1}{2}\right)}$$

$$= t^{a(\mathbb{G})}\, t^{\left(\frac{1-n}{2}\right)} \overbrace{\sum_{x} (-1)^{M(x)}\, t^{-\sum' w_{\mathbb{D}}(x)}}$$

Proof (cont.):

$$= t^{a(\mathbb{G})} \underbrace{t^{\left(\frac{1-n}{2}\right)} (-1)^{n-1}} \det(M(\mathbb{G})) \, \varepsilon(\mathbb{G})$$

$$= \underline{t}^{a(\mathbb{G})} (1-t^{-1})^{n-1} \underline{(t^{1/2}-t^{-1/2})^{1-n} \det(M(\mathbb{G})) \, \varepsilon(\mathbb{G})}$$

$$= (1-t^{-1})^{n-1} D_{\mathbb{G}}(t)$$

$$= (1-t^{-1})^{n-1} \cdot \Delta_K$$

Theorem (4.7.6)

The graded Euler characteristic of the simply blocked grid homology is equal to the (symmetrized) Alexander polynomial $\Delta_K(t)$:

$$\chi(\widehat{GH}(\mathbb{G})) = \Delta_K(t).$$

Proof.

$$\widetilde{GH}(\mathbb{G}) \cong \widehat{GH}(\mathbb{G}) \otimes W^{\otimes n-1}$$

$$\chi(\widetilde{GH}(\mathbb{G})) = \chi(\widehat{GH}(\mathbb{G}))\, \chi(W)^{n-1} \quad (*)$$

$$\chi(\widetilde{GH}(\mathbb{G})) = (1-t^{-1})^{n-1} \cdot \Delta_K(t) \quad (1)$$

$$\chi(W) = \sum_{d,s} (-1)^d \dim W_{d,s}\, t^s = 1 - t^{-1} \quad (2)$$

$$(1-t^{-1})^{n-1} \cdot \Delta_K = \chi(\widehat{GH}(\mathbb{G}))\, (1-t^{-1})^{n-1} \Rightarrow \chi(\widehat{GH}(\mathbb{G})) = \Delta_K \quad \square$$

A, B

$$\chi(A \otimes B) = \chi(A) \cdot \chi(B)$$

Exercises

Exercise 1. Consider the grid diagram of size 5×5 for the trefoil knot in the figure with grid state $\mathbf{x}$ denoted in red. For each $x \in \mathbf{x}$, verify that $w_{\mathcal{D}}(x) = \mathcal{J}(x, \mathbb{O} - \mathbb{X})$.

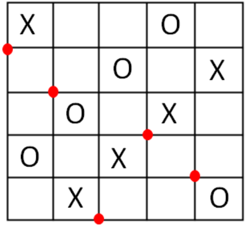


Figure: Grid diagram for the trefoil knot with a grid state $\mathbf{x}$

Exercises

Exercise 2. Compute the Alexander grading of the grid state **x** in the figure by means of the formula

$$A(x) = A'(x) + a(\mathbb{G}) - \left(\frac{n-1}{2}\right).$$

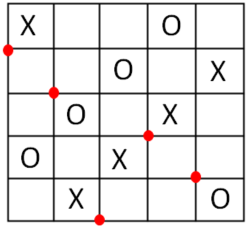


Figure: Grid diagram for the trefoil knot with a grid state **x**

$$\chi(\widehat{GH}(k)) = \sum_{d,s} (-1)^d \dim GH_d(k,s)\, t^s$$

$$\begin{cases} \mathbb{F} & (d,s) \in \{(0,1), (-1,0), (2,-1)\} \\ 0 & \text{otherwise} \end{cases}$$

$$= (-1)^0 \dim GH_0(k,1)\, t$$

$$+ (-1)^{-1} \dim GH_{-1}(k,0)\, t^0$$

$$+ (-1)^2 \dim GH_2(k,-1)\, t^{-1}$$

(Exercise 4)

$$= t - 1 + t^{-1}$$

Exercises

Exercise 3. Show that the (graded) Euler characteristic of a chain complex is equal to the (graded) Euler characteristic of its homology.

$$\to C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} C_{i-2} \xrightarrow{\partial_{i-2}} C_{i-3} \xrightarrow{\partial_{i-3}} \cdots$$

$$0 \to C_2 \to C_1 \to C_0 \to 0$$

$$\frac{\ker(\partial_i)}{Im(\partial_{i+1})}$$

$$\uparrow$$

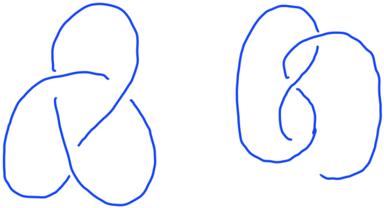
$$\sum_i (-1)^i \dim(H_i(C)) = \chi$$

$$\dim(H_i(C)) = \dim(\ker(\partial_i)) - \dim(im(\partial_{i+1}))$$

$$\dim(C_i) = \dim(\ker(\partial_i)) + \dim(im(\partial_i))$$

Exercises

Exercise 4. By computing the Alexander polynomial of the trefoil knot and the Euler characteristic of the simply blocked grid homology of the trefoil, verify that they are equal.



$$\Delta = \det(S t^{-1/2} - S^T t^{1/2})$$

$$S = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$= \dots = t - 1 + t^{-1}$$

$$\widehat{GH}_d(K,s) = \begin{cases} \mathbb{F} & \text{if } (d,s) \in \{(0,1), (-1,0), (2,-1)\} \\ 0 & \text{otherwise} \end{cases}$$

Exercise 5. Let $X = \bigoplus_{d,s} X_{d,s}$ and $Y = \bigoplus_{d,s} Y_{d,s}$ be two bigraded vector spaces. Prove that

$$\chi(X \otimes Y) = \chi(X) \cdot \chi(Y).$$

$$X = \bigoplus_d X_d, \quad Y = \bigoplus_d Y_d$$

$$(X \otimes Y)_d = \bigoplus_{d_1+d_2=d} X_{d_1} \otimes Y_{d_2}$$

$$\chi(X \otimes Y) = \sum_d (-1)^d \dim (X \otimes Y)_d = \sum_d (-1)^d \dim \left(\bigoplus_{d_1+d_2=d} X_{d_1} \otimes Y_{d_2} \right)$$

$$= \sum_d (-1)^d \sum_{d_1+d_2=d} \dim(X_{d_1}) \dim(Y_{d_2})$$

Exercise 6. (Part of the proof of Lemma 6.) Let $\mathbb{G}$ denote a grid diagram and consider a square with center z that is marked with O. There is one X marking in the same row as the square with center z, call this marking X_1 and there is one X marking in the same column as the square with center z, call this marking X_2. Let $\mathbb{O}'$ denote the set of all markings in $\mathbb{O}$ different from O. Let $\mathbb{X}'$ denote all the markings in $\mathbb{X}$ different from X_1 and X_2. By consider all different cases, prove the following statements.

(a) For any $O' \in \mathbb{O}'$, it holds that

$$\mathcal{J}(z, O') = \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, O').$$

(b) For any $X' \in \mathbb{X}'$, it holds that

$$\mathcal{J}(z, X') = \frac{1}{4}\mathcal{J}(z_1 + z_2 + z_3 + z_4, X').$$

Exercise 5

$$\chi(X)\chi(Y) = \sum_{d_1}(-1)^{d_1}\dim(X_{d_1}) \sum_{d_2}(-1)^{d_2}\dim(Y_{d_2})$$

$$= \square$$

References I

Peter S Ozsváth, András I Stipsicz, and Zoltán Szabó.
***Grid homology for knots and links*, volume 208.**
American Mathematical Soc., 2017.

Extra frame

$\dim(H_i(C)) = \dim(\ker(\partial_i)) - \dim(\operatorname{im}(\partial_{i-1}))$ (exercise 3)

$\dim(C_i) = \dim(\ker(\partial_i)) + \dim(\operatorname{im}(\partial_i))$

$-\dim(H_1(C)) + \dim(H_2(C)) + \dim(H_0(C)) =$

$-\ker(\partial_1) + \operatorname{im}(\partial_2) + \ker(\partial_2) + \dim(C_0) - \dim(\operatorname{im}(\partial_1)) =$

$\dim(C_2) - \dim(C_1) + \dim(C_0)$