Introduction

The purpose of this seminar will be to complete some calculations (tip to toe) pertaining to the homologies of some knots. In particular, we shall calculate the unblocked and simply blocked grid homologies of the unknot and the right-handed trefoil knot. This will be accomplished mainly by juggling applicable propositions from the book. Furthermore, as a conclusion we mention the result that these homologies are knot invariants, in one form or another. We end the document with some exercises for the (interested) reader.

1 Regarding the unknot

For the unknot, we rely on the diagram as presented in Figure 1. In this diagram,

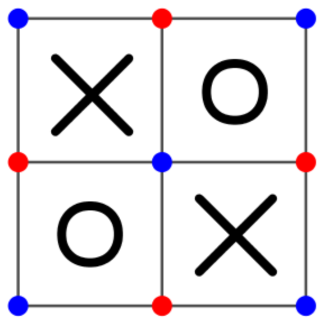


Figure 1: Toroidal grid diagram $\mathbb{G}$ of the unknot $\mathcal{O}$, with two grid states $\boldsymbol{x}$ and $\boldsymbol{y}$.

we distinguish the only two grid states $\boldsymbol{x}$ and $\boldsymbol{y}$. As a warming up, we show that the differential $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ is in fact zero for both these states:

$$\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\boldsymbol{x}) = \sum_{\boldsymbol{s}\in\{\boldsymbol{x},\boldsymbol{y}\}} \mathbf{1}_{\boldsymbol{x},\boldsymbol{s}}\boldsymbol{s} = \mathbf{1}_{\boldsymbol{x},\boldsymbol{x}}\boldsymbol{x} + \mathbf{1}_{\boldsymbol{x},\boldsymbol{y}}\boldsymbol{y} = 0 \times \boldsymbol{x} + 0 \times \boldsymbol{y} = 0, \tag{1}$$

where we used Albert's indicator function notation,

$$\mathbf{1}_{\boldsymbol{a},\boldsymbol{b}} := \#\{r \in \mathrm{Rect}^\circ(\boldsymbol{a},\boldsymbol{b}) : r \cap \mathbb{O} = r \cap \mathbb{X} = \emptyset\}. \tag{2}$$

and we realized that $\mathbf{1}_{\boldsymbol{x},\boldsymbol{x}} = 0$ as there are no empty rectangles from $\boldsymbol{x}$ to itself, and $\mathbf{1}_{\boldsymbol{x},\boldsymbol{y}} = 0$ as every rectangle must contain either an $\times$ or $\circ$. Evidently, a reasoning similar to equation (1) holds for $\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\boldsymbol{y})$.

This is not too interesting. Instead, we elect to investigate the *unblocked grid homology* $GH^-(\mathbb{G})$, which is defined similarly to the blocked grid homology.

Notation 1. We write

$$r \in \mathrm{Rect}^\circ_{\mathbb{X}}(\boldsymbol{a},\boldsymbol{b}) \iff r \in \mathrm{Rect}^\circ(\boldsymbol{a},\boldsymbol{b}) \quad \text{and} \quad r \cap \mathbb{X} = \emptyset. \tag{3}$$

Notation 2. We write $GH_{d,s} := GH_d(\mathbb{G}, s)$, where we can either refer to the tilde, hat, or minus "flavors," and the grid $\mathbb{G}$ we are talking about is presumed known. A similar notation holds for the grid complexes GC.

Definition 3. (4.6.1 of the book) The *unblocked grid complex* GC^- is the free module over $\mathbb{F}[V_1, \dots, V_n]$ generated by $\mathbf{S}(\mathbb{G})$, where n is the grid number. The associated differential $\partial_{\mathbb{X}}^-$ acts on a grid state $\boldsymbol{s}$ as

$$\partial_{\mathbb{X}}^-(\boldsymbol{s}) = \sum_{\boldsymbol{t}\in\mathbf{S}(\mathbb{G})} \sum_{r\in\mathrm{Rect}^\circ_{\mathbb{X}}(\boldsymbol{s},\boldsymbol{t})} \left(\prod_{i=1}^{n} V_i^{O_i(r)}\right) \boldsymbol{t}, \tag{4}$$

where $O_i(r)$ is an indicator function for the ith $\circ$ marking:

$$O_i(r) = \begin{cases} 1 & \text{if} \quad \circ_i \in r \\ 0 & \text{otherwise} \end{cases}, \tag{5}$$

i.e. it is one if the ith $\circ$ is in r, and zero otherwise.

As one can see, the unblocked grid complex allows us to bring fields into the mix, and as such introduce some degrees of freedom (and more interesting math!). The complex together with the differential can form a homology, as stated in the next definition.

Definition 4. (4.6.11 of the book) The *unblocked grid homology* of some grid diagram $\mathbb{G}$, $GH^-(\mathbb{G})$, is defined as the homology of $(GC^-(\mathbb{G}), \partial_{\mathbb{X}}^-)$., when considered as a module over $\mathbb{F}[U]$, with U a placeholder for one of the V_i.

The justifications of the necessary conditions for this definition to make sense (e.g. that $(\partial_{\mathbb{X}}^-)^2 \equiv 0$) are omitted from this text; they can be found in the material of chapter 4.6 prior to Definition 4.6.11. In order to find GH^- for the unknot, we first need to find the relevant complex(es) $GC^-(\mathcal{O})$, i.e. the Alexander and Maslov gradings of $\boldsymbol{x}$ and $\boldsymbol{y}$ – this is done in Lemma 5. And, we need to investigate how the differential $\partial_{\mathbb{X}}^-$ acts on the states $\boldsymbol{x}$ and $\boldsymbol{y}$, which is done in Lemma 6 Then, we proceed with finding out the unblocked grid homology in Proposition 7.

Lemma 5. *For the grid states $\boldsymbol{x}$ and $\boldsymbol{y}$, we have $A(\boldsymbol{x}) = M(\boldsymbol{x}) = -1$ and $A(\boldsymbol{y}) = M(\boldsymbol{y}) = 0$.*

Proof. The proof is by straightforward calculation, using the formulae for A and M as found in section 4.3 of the book. Observe that

$$M_{\mathbb{O}}(\boldsymbol{x}) = \mathcal{J}(\boldsymbol{x},\boldsymbol{x}) + \mathcal{J}(\mathbb{O},\mathbb{O}) - 2\mathcal{J}(\boldsymbol{x},\mathbb{O}) + 1 = 3 + 1 - 2\cdot 3 + 1 = -1 \tag{6}$$

and similarly $M_{\mathbb{X}}(\boldsymbol{x}) = 0$ (swap $\mathcal{J}(\mathbb{O},\mathbb{O}) \leftrightarrow \mathcal{J}(\mathbb{X},\mathbb{X}) = 0$ and $\mathcal{J}(\boldsymbol{x},\mathbb{O}) \leftrightarrow \mathcal{J}(\boldsymbol{x},\mathbb{X}) = 2$). As such,

$$A(\boldsymbol{x}) = \tfrac{1}{2}(M_{\mathbb{O}}(\boldsymbol{x}) - M_{\mathbb{X}}(\boldsymbol{x})) - (n-1)/2 = -1. \tag{7}$$

Furthermore, for $\boldsymbol{y}$, we have

$$M_{\mathbb{O}}(\boldsymbol{y}) = \mathcal{J}(\boldsymbol{y},\boldsymbol{y}) + \mathcal{J}(\mathbb{O},\mathbb{O}) - 2\mathcal{J}(\boldsymbol{y},\mathcal{O}) + 1 = 2 + 1 - 2\cdot 2 + 1 = 0, \tag{8}$$

and similarly $M_{\mathbb{X}}(\boldsymbol{y}) = -1$ (swap $\mathcal{J}(\mathbb{O},\mathbb{O}) \leftrightarrow \mathcal{J}(\mathbb{X},\mathbb{X}) = 0$ and $\mathcal{J}(\boldsymbol{y},\mathbb{O}) \leftrightarrow \mathcal{J}(\boldsymbol{y},\mathbb{X}) = 2$). Then,

$$A(\boldsymbol{y}) = \tfrac{1}{2}(M_{\mathbb{O}}(\boldsymbol{y}) - M_{\mathbb{X}}(\boldsymbol{y})) - (n-1)/2 = 0, \tag{9}$$

and the computation is complete. □

Lemma 6. *We have that $\partial^-_{\mathbb{X}}$ acts on the states $\boldsymbol{x}$ and $\boldsymbol{y}$ as:*

$$\partial^-_{\mathbb{X}}(\boldsymbol{x}) = (V_1 + V_2)\boldsymbol{y}, \quad \textit{and} \quad \partial^-_{\mathbb{X}}(\boldsymbol{y}) = 0. \tag{10}$$

Proof. Letting $\partial^-_{\mathbb{X}}$ act upon $\boldsymbol{x}$, we have

$$\begin{aligned}
\partial^-_{\mathbb{X}}(\boldsymbol{x}) &= \sum_{\boldsymbol{s}\in\{\boldsymbol{x},\boldsymbol{y}\}} \sum_{r\in\mathrm{Rect}^\circ_{\mathbb{X}}(\boldsymbol{x},\boldsymbol{s})} V_1^{O_1(r)} V_2^{O_2(r)} \boldsymbol{s} \\
&= \sum_{r\in\mathrm{Rect}^\circ_{\mathbb{X}}(\boldsymbol{x},\boldsymbol{x})} V_1^{O_1(r)} V_2^{O_2(r)} \boldsymbol{x} + \sum_{r\in\mathrm{Rect}^\circ_{\mathbb{X}}(\boldsymbol{x},\boldsymbol{y})} V_1^{O_1(r)} V_2^{O_2(r)} \boldsymbol{y} \\
&\overset{\star}{=} (V_1^{O_1(r_1)} V_2^{O_2(r_1)} + V_1^{O_1(r_2)} V_2^{O_2(r_2)})\boldsymbol{y} \\
&= (V_1 + V_2)\boldsymbol{y}
\end{aligned}$$

where for $\star$ we used that the sum over $\mathrm{Rec}^\circ_{\mathbb{X}}(\boldsymbol{x},\boldsymbol{x})$ is zero (for it's a sum over an empty set). The last step references rectangles r_1 and r_2: these are the lower left and upper right squares of Figure 1. Since both of them contain precisely one $\circ$-marking, they will light up one of the indicator functions O_1, O_2, giving the equation its final form.

Then, to find $\partial^-_{\mathbb{X}}(\boldsymbol{y})$, we need to sum over the sets $\mathrm{Rect}^\circ_{\mathbb{X}}(\boldsymbol{y},\boldsymbol{x})$ and $\mathrm{Rect}^\circ_{\mathbb{X}}(\boldsymbol{y},\boldsymbol{y})$. These are both empty though: the latter trivially so, and the former because every rectangle from $\boldsymbol{y}$ to $\boldsymbol{x}$ contains an $\times$. As such, we have $\partial^-_{\mathbb{X}}(\boldsymbol{y}) = 0$, and the proof is complete. □

Proposition 7. *(4.8.1 of the book) For the unknot $\mathcal{O}$, we have that $GH^-(\mathcal{O}) \cong \mathbb{F}[U]$.*

Proof. We identify that by Lemma 5, the complex $GC^-_{-1,-1}$ is generated by $\boldsymbol{x}$, and $GC^-_{0,0}$ is generated by $\boldsymbol{y}$, both over $\mathbb{F}[V_1,V_2]$. Furthermore, these are the only two complexes that we need to consider (there are no other generators). By Lemma 6, then, we have that $\ker(\partial^-_{\mathbb{X}}) = \mathbb{F}[V_1,V_2]\boldsymbol{y}$ and $\mathrm{Im}(\partial^-_{\mathbb{X}}) = \mathbb{F}[V_1,V_2](V_1+V_2)\boldsymbol{y}$. We can now compute the partial homologies:

$$GH^-_{-1,-1} = \frac{\ker(\partial^-_{\mathbb{X}}) \cap GC^-_{-1,-1}}{\mathrm{Im}(\partial^-_{\mathbb{X}}) \cap GC^-_{-1,-1}} = 0 \tag{11}$$

since both intersections are trivial. Therefore, we can already promote the homology of the grading $d,s = 0$, $GH^-_{0,0}$, to be the full homology of the unknot, $GH^-(\mathcal{O})$. We then find

$$GH^-(\mathcal{O}) = \frac{\ker(\partial^-_{\mathbb{X}})}{\mathrm{Im}(\partial^-_{\mathbb{X}})} = \frac{\mathbb{F}[V_1,V_2]\boldsymbol{y}}{\mathbb{F}[V_1,V_2](V_1+V_2)\boldsymbol{y}} \cong \frac{\mathbb{F}[V_1,V_2]}{\mathbb{F}[V_1,V_2](V_1+V_2)} \cong \mathbb{F}[U]_{(0,0)}, \tag{12}$$

where U is a placeholder for one of the variables and the subscript $(0,0)$ on $\mathbb{F}[U]$ indicates the bigrading. Therefore, this proposition is proven. $\square$

As can be seen from the above, the unblocked grid homology has the potential to escalate rather quickly, with the amount of its variables V_i growing with grid size. As such, it may also prove fruitful to investigate a slightly simpler flavor of the grid homology.

Definition 8. (4.6.12 of the book) For some $i = 1, \dots, n$, the *simply blocked grid complex* $\widehat{GC}(\mathbb{G})$ is defined as $\widehat{GC}(\mathbb{G}) := GC^-(\mathbb{G})/V_i$. The resulting *simply blocked grid homology* is defined $\widehat{GH}(\mathbb{G}) := (\widehat{GC}(\mathbb{G}), \partial_{\mathbb{X}}^-)$.

Proposition 9. *(4.8.1 of the book) For the unknot $\mathcal{O}$, we have that $\widehat{GH}(\mathcal{O}) \cong \mathbb{F}$.*

Proof. According to definition, we may mod out one of the variables of $\mathbb{F}[V_1, V_2]$ over which $GC^-(\mathcal{O})$ is generated – as such, set $V_2 \equiv 0$. Then, our differential now acts on the states as

$$\partial_{\mathbb{X}}^-(\boldsymbol{x}) = V_1 \boldsymbol{y}, \quad \text{and} \quad \partial_{\mathbb{X}}^-(\boldsymbol{y}) = 0, \tag{13}$$

and accordingly $\ker(\partial_{\mathbb{X}}^-) = \mathbb{F}[V_1]\boldsymbol{y}$ and $\mathrm{Im}(\partial_{\mathbb{X}}^-) = \mathbb{F}[V_1]V_1\boldsymbol{y}$. Thus,

$$\widehat{GH}(\mathcal{O}) = \frac{\ker(\partial_{\mathbb{X}}^-)}{\mathrm{Im}(\partial_{\mathbb{X}}^-)} = \frac{\mathbb{F}[V_1]\boldsymbol{y}}{\mathbb{F}[V_1]V_1\boldsymbol{y}} \cong \frac{\mathbb{F}[V_1]}{\mathbb{F}[V_1]V_1} \cong \mathbb{F}, \tag{14}$$

which completes the proof. $\square$

2 Regarding the trefoil knot

The finding of the above defined homologies isn't limited to the unknot only; they can be found for other knots as well. In this section we shall consider the right-handed trefoil knot (i.e. $T_{2,3}$), and attempt to find its unblocked and simply blocked homologies, based on the material in section 4.8. One may find the grid diagram of the trefoil knot (and the associated winding numbers to each grid vertex) in Figure 2. One item of note is that the winding numbers depicted in this figure are at the same time the winding numbers as meant in Definition 3.3.1 *and* as meant in Definition 4.7.1. This latter definition (which is more a recharacterization – see the book) is what we particularly require for the following lemma.

Lemma 10. *In the right-handed trefoil $T_{2,3}$, there is no grid state with Alexander grading greater than 1.*

Proof. For the proof of the lemma, we utilize the result of Proposition 4.7.2 of the book, being

$$A(\boldsymbol{x}) = A'(\boldsymbol{x}) + a(\mathbb{G}) - \frac{n-1}{2}, \quad \text{where} \quad A'(\boldsymbol{x}) := -\sum_{p \in \boldsymbol{x}} w_{\mathcal{D}}(p), \tag{15}$$

i.e. $A'(\boldsymbol{x})$ is the negative sum of winding numbers around all points in the grid state $\boldsymbol{x}$. By inspection, we verify that $a(\mathbb{G}) = -3$ and $(n-1)/2 = 2$, so in fact $A(\boldsymbol{x}) = A'(\boldsymbol{x}) - 5$.

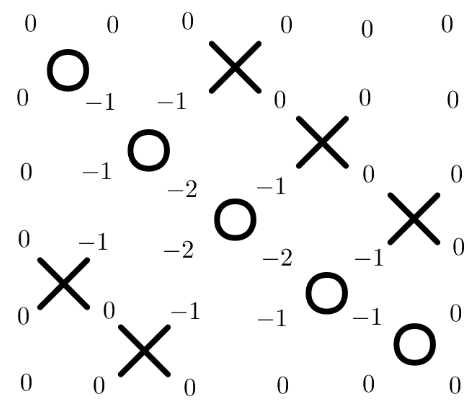


Figure 2: The right-handed trefoil knot $T_{2,3}$, with winding numbers filled in.

Now, the proof of the lemma comes down to showing that $A'(\boldsymbol{x}) \leq 6$ for all possible grid states. Since the middle square (referring to Figure 2) has all the possible contributions, we can only effectively pick *four* grid states. Moreover, only *two* of these can have the contribution -2: there are only three -2 points, and picking all three at the same time is illegal (due to colinearity). If we pick two of our grid states to have contribution -2, then the other two can have a maximum contribution of -1 each. As such, the total sum of contributions cannot reach below -6, meaning that $A'(\boldsymbol{x})$ cannot be greater than 6.

Since we have now shown $A'(\boldsymbol{x}) \leq 6$, it follows that $A(\boldsymbol{x}) \leq 1$ and the lemma is proven. □

This means that we do not have to consider complexes which have Alexander grading greater than 1, already slashing some complexity. Furthermore, we shall need the following proposition:

Proposition 11. *(4.6.15 of the book) We have that*

$$\begin{aligned}\widetilde{GH}_{d,s} &\cong \widehat{GH}_{d,s} \otimes W^{\otimes(n-1)} \\ &\overset{n=5}{\cong} \widehat{GH}_{d,s} \oplus \widehat{GH}^{\oplus 4}_{d+1,s+1} \oplus \widehat{GH}^{\oplus 6}_{d+2,s+2} \oplus \widehat{GH}^{\oplus 4}_{d+3,s+3} \oplus \widehat{GH}_{d+4,s+4},\end{aligned}$$

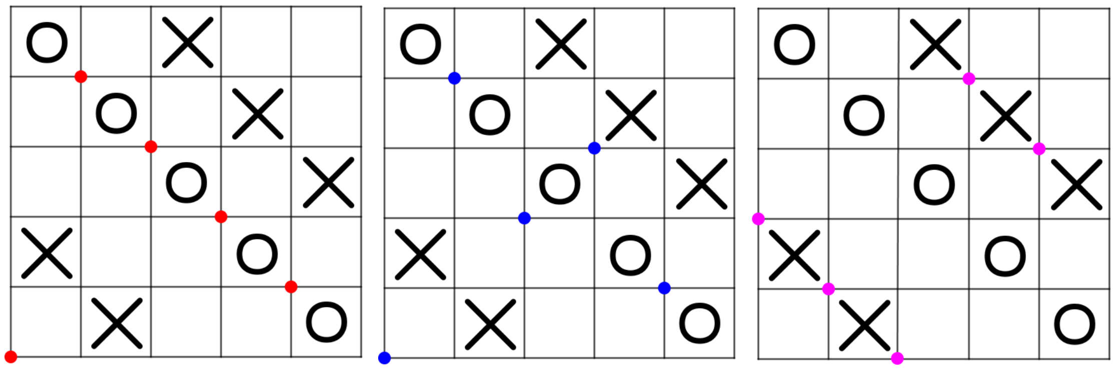
where W acts as $\widehat{GH}_{d,s} \otimes W \cong \widehat{GH}_{d,s} \oplus \widehat{GH}_{d+1,s+1}$.

Proof. Please see the book (and preceding portion of text) for a proof of this. □

At once, we may combine the two above results for the case where the Alexander grading $s = 1$.

Proposition 12. *For some grid state $\boldsymbol{x}$ on $T_{2,3}$, let $A(\boldsymbol{x}) = s = 1$. Then, the Maslov grading $d = 0$ and we have*

$$\widehat{GH}_{0,1} \cong \widetilde{GH}_{0,1} \cong \mathbb{F} \tag{16}$$



(a) State $\boldsymbol{a}$: $A(\boldsymbol{a}) = 1$ and $M(\boldsymbol{a}) = 0$.
(b) State $\boldsymbol{b}$: $A(\boldsymbol{b}) = 0$ and $M(\boldsymbol{b}) = -1$.
(c) State $\boldsymbol{c}$: $A(\boldsymbol{c}) = -5$ and $M(\boldsymbol{c}) = -6$.

Figure 3: Some grid states on the trefoil $T_{2,3}$.

Proof. We are actually being a little cheeky: there exists exactly one grid state on $T_{2,3}$ which has $s = 1$ – this is $\boldsymbol{a}$ c.f. Figure 3. Its Maslov grading is $d = 0$ (verified by inspection), and so the first part of the lemma is shown.

Then, we need to find the intersections of the image and kernel of $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ with $\widetilde{GC}_{0,1}$. For one, notice that since there is no state with $d, s = 1$, we have that $\mathrm{Im}(\tilde{\partial}_{\mathbb{O},\mathbb{X}}) \cap \widetilde{GC}_{0,1} = 0$. Furthermore, we claim that $\boldsymbol{a}$ is in the kernel of $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$: indeed, with some thought we see that any rectangle consisting of precisely two points of $\boldsymbol{a}$ has at least one $\circ$ or $\times$ in it, so that $\mathbf{1}_{\boldsymbol{a},\boldsymbol{s}} = 0$ for any other relevant grid state $\boldsymbol{s}$. Thus, $\ker(\tilde{\partial}_{\mathbb{O},\mathbb{X}}) \cap \widetilde{GC}_{0,1} = \mathbb{F}\boldsymbol{a}$, and we can now write

$$\widetilde{GH}_{0,1} = \frac{\ker(\tilde{\partial}_{\mathbb{O},\mathbb{X}}) \cap \widetilde{GC}_{0,1}}{\mathrm{Im}(\tilde{\partial}_{\mathbb{O},\mathbb{X}}) \cap \widetilde{GC}_{0,1}} = \mathbb{F}\boldsymbol{a} \cong \mathbb{F}. \tag{17}$$

Then, we employ Proposition 11. Namely, we observe that since there exist no grid states with $s > 1$, it stands to reason that $\widehat{GH}_{d,s>1}$ is in fact the trivial vector space, and we may disregard it from calculation. As such, its result boils down to $\widehat{GH}_{0,1} \cong \widetilde{GH}_{0,1}$, which when combined with equation (17) then yields the desired result. □

In the above proposition, we were lucky to some degree: there existed precisely one state, being $\boldsymbol{a}$, such that $s = 1$ and $d = 0$. However, when we move to different combinations of these gradings, it becomes a whole lot harder to see whether we have found *all* the states generating the complex with that particular bigrading. To this end, we call upon the power of computers. In Table 1, we have recorded for every allowed grid state on $T_{2,3}$ its Alexander and Maslov gradings. The power of the table will become visible at once in the next proposition.

Proposition 13. *We have the following statements:*

1. $\widehat{GH}_{-1,0} \cong \mathbb{F}$
2. $\widehat{GH}_{-2,-1} \cong \mathbb{F}$

d \ s	-5	-4	-3	-2	-1	0	1
-6	1	0	1	0	0	0	0
-5	0	5	5	0	0	0	0
-4	0	0	20	6	1	0	0
-3	0	0	5	30	5	0	0
-2	0	0	0	10	20	0	0
-1	0	0	0	0	5	5	0
0	0	0	0	0	0	0	1

Table 1: Amount of states with given Alexander and Maslov gradings s and d. The information was generated using the code in Listing 1.

3. $\widehat{GH}_{d,s} = 0$ *for every other combination of* d *and* s*.*

Proof. We shall prove point by point.

1. From Table 1, we see that there are precisely five states which have $s = 0$ and $d = -1$ – as such, the complex $\widetilde{GC}_{-1,0}$ is generated by these five states over $\mathbb{F}$, and $\widetilde{GC}_{-1,0} \cong \mathbb{F}^{\oplus 5}$.

   Now observe that since by differentiation all states are sent to 0 (as there are no states with $s = 0$ and $d = -2$, by Table 1), we have that all five states are in the kernel of $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$. At the same time, since there are no states that map upon states with $s = 0$ and $d = -1$ (again, from Table 1), we conclude that the image of $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ intersected with the complex is empty. As such, we conclude that $\widetilde{GH}_{-1,0} = \widetilde{GC}_{-1,0} \cong \mathbb{F}^{\oplus 5}$.

   Then, invoke Proposition 11 to say that

$$\widetilde{GH}_{-1,0} \cong \mathbb{F}^{\oplus 5} \cong \widehat{GH}_{-1,0} \oplus \widehat{GH}_{0,1}^{\oplus 4} \cong \widehat{GH}_{-1,0} \oplus \mathbb{F}^{\oplus 4}, \tag{18}$$

   where we omitted "higher order" terms of $\widehat{GH}$, as these are trivial (there are no states with sufficiently high Alexander grading). From equation (18), then, we conclude that $\widehat{GH}_{-1,0} \cong \mathbb{F}$, and this point is proven.

2. Observe that, by a reasoning as above, we have that $\widetilde{GH}_{-6,-5} \cong \mathbb{F}$ and $\widetilde{GH}_{-5,-4} \cong \mathbb{F}^{\oplus 5}$. But, these also feature in the results of Proposition 11:

$$\widetilde{GH}_{-6,-5} \cong \mathbb{F} \cong \widehat{GH}_{-6,-5} \oplus \widehat{GH}_{-5,-4}^{\oplus 4} \oplus \widehat{GH}_{-4,-3}^{\oplus 6} \oplus \widehat{GH}_{-3,-2}^{\oplus 4} \oplus \widehat{GH}_{-2,-1} \tag{19}$$

   and

$$\widetilde{GH}_{-5,-4} \cong \mathbb{F}^{\oplus 5} \cong \widehat{GH}_{-5,-4} \oplus \widehat{GH}_{-4,-3}^{\oplus 4} \oplus \widehat{GH}_{-3,-2}^{\oplus 6} \oplus \widehat{GH}_{-2,-1}^{\oplus 4} \oplus \underbrace{\widehat{GH}_{-1,0}}_{\cong \mathbb{F}}. \tag{20}$$

Then, to unify these equations – and under the reasonable assumption that $\widehat{GH}_{d,s} \cong \mathbb{F}^{\oplus \ell}$ for some $\ell \in \mathbb{N} \cup \{0\}$ – we must conclude that $\widehat{GH}_{-5,-4} = \widehat{GH}_{-4,-3} = \widehat{GH}_{-3,-2} = 0$: indeed, if they had any copy of $\mathbb{F}$, equation (19) would be violated. But then to satisfy equation (20), we must conclude that $\widehat{GH}^{\oplus 4}_{-2,-1} \cong \mathbb{F}^{\oplus 4} \implies \widehat{GH}_{-2,-1} \cong \mathbb{F}$. Thus (and with the quick sanity check that this also satisfies equation (19) with the additional result that $\widehat{GH}_{-6,-5} = 0$), this point is proven also.

3. We shall prove this in a couple of different points.
   (a) We recognize that by part 2, $\widehat{GH}_{-6,-5} = \cdots = \widehat{GH}_{-3,-2} = 0$.
   (b) Observe that $\widetilde{GH}_{-5,-5} = 0$, owing to the fact that there are no states with $s = d = -5$, so $\widetilde{GC}_{-5,-5} = 0$. However, we also know by Proposition 11,
   $$\widetilde{GH}_{-5,-5} = 0 \cong \widehat{GH}_{-5,-5} \oplus \widehat{GH}^{\oplus 4}_{-4,-4} \oplus \widehat{GH}^{\oplus 6}_{-3,-3} \oplus \widehat{GH}^{\oplus 4}_{-2,-2} \oplus \widehat{GH}_{-1,-1}. \tag{21}$$
   As such, we conclude that $\widehat{GH}_{-5,-5} = \cdots = \widehat{GH}_{-1,-1} = 0$.
   (c) Furthermore, we have $\widetilde{GH}_{-6,-4} = 0$, owing once more to the fact that there are no states with $d = -4$ and $s = -6$. And since, by Proposition 11,
   $$\widetilde{GH}_{-6,-4} = 0 \cong \widehat{GH}_{-6,-4} \oplus \widehat{GH}^{\oplus 4}_{-5,-3} \oplus \widehat{GH}^{\oplus 6}_{-4,-2} \widehat{GH}^{\oplus 4}_{-3,-1} \oplus \widehat{GH}_{-2,0}, \tag{22}$$
   we must conclude that $\widehat{GH}_{-6,-4} = \cdots = \widehat{GH}_{-2,0} = 0$.
   (d) That $\widehat{GH}_{-6,-3} = \widehat{GH}_{-4,-1} = 0$ is left as an exercise.

Since all the points have been proven, this proposition is proven now also. $\square$

Notice that the combination of Propositions 12 and 13 leads us to the conclusion

$$\widehat{GH}_{d,s} = \begin{cases} \mathbb{F} & \text{if} \quad (d,s) \in \{(0,1), (-1,0), (-2,-1)\} \\ 0 & \text{otherwise} \end{cases}, \tag{23}$$

which completes Exercise 4.8.2(a) from the book. Now, what about the minus flavor of the homology? We shall need to make use of the following proposition, which connects GH^- to $\widehat{GH}$.

Proposition 14. *(4.6.18 of the book) We have that the following is a long exact sequence:*

$$\cdots \longrightarrow GH^-_{d+2,s+1} \xrightarrow{U} GH^-_{d,s} \longrightarrow \widehat{GH}_{d,s} \longrightarrow GH^-_{d+1,s+1} \longrightarrow \dots \tag{24}$$

Proof. Please see the proof in the book. $\square$

Together with the full information that we have about $\widehat{GH}_{d,s}$, we should be able to find the GH^- as well. And indeed, we can: this is content of the following proposition.

Proposition 15. *We have that the following statements hold.*

1. *We have* $GH^-_{d,s} = \mathbb{F}$ *for* $(d,s) \in \{(0,1),(-2,-1)\}$.
2. *Moreover,* $GH^-_{d,s} = \mathbb{F}$ *for* $(d,s) = (-2k,-k)$ *for all* $k \in \mathbb{N}$.
3. *Finally,* $GH^-_{d,s} = 0$ *otherwise.*

Proof. We shall prove point for point.

1. To see that $GH^-_{0,1} \cong \mathbb{F}$, we note that $GC^-_{d,s}$ is spanned by the vectors $V_1^{k_1} \dots V_n^{k_n} \boldsymbol{x}$, where $\{k_i\}_{i=1}^n$ (non-negative integers) and $\boldsymbol{x}$ must be such that

$$\begin{cases} d = M(\boldsymbol{x}) - 2\sum_i k_i \\ s = A(\boldsymbol{x}) - \sum_i k_i \end{cases}, \tag{25}$$

which can be found at the top of page 76 of the book. Given that the maximum Alexander grading of any state on $T_{2,3}$ is already 1, we must conclude that for the case $(d,s) = (0,1)$, $k_i = 0$ for all i for the above requirements to have any chance of holding. Then additionally realizing that there is *one* state with $(d,s) = (0,1)$ – being our friend $\boldsymbol{a}$ – we conclude that $GC^-_{0,1}$ is generated over $\mathbb{F}$ by $\boldsymbol{a}$, so that $GC^-_{0,1} \cong \mathbb{F}$. Finally, since $\partial^-_{\mathbb{X}}$ has homogeneity $(-2,-1)$ and there are no states with either $(d,s) = (-2,0)$ or $d,s = (2,2)$, by reasoning that we have seen we conclude that $GH^-_{0,1} \cong \mathbb{F}$ as well.

For $GH^-_{-2,-1} \cong \mathbb{F}$, consider equation (25) again, but now for $(d,s) = (0,0)$. Then, we must have that $A(\boldsymbol{x}) = \sum_i k_i$ and $M(\boldsymbol{x}) = 2\sum_i k_i$. But then,

$$A(\boldsymbol{x}) \leq 1 \implies A(\boldsymbol{x}) = \sum_i k_i = 0 \quad \text{or} \quad 1 \implies M(\boldsymbol{x}) = 0 \quad \text{or} \quad 2$$
$$\overset{\star}{\implies} \sum_i k_i = A(\boldsymbol{x}) = M(\boldsymbol{x}) = 0,$$

where $\star$ holds as there are no states with $M(\boldsymbol{x}) = 2$. This means that $GC^-_{0,0}$ is generated by states with $(d,s) = (0,0)$. But, since there are no such states, we conclude that $GC^-_{0,0} = GH^-_{0,0} = 0$.

We can repeat much the same reasoning when considering $GC^-_{-1,0}$: this is generated with $\sum_i k_i = 0$ by the five states with $(d,s) = (-1,0)$, so $GC^-_{-1,0} \cong \mathbb{F}^{\oplus 5}$. But now, since the differential $\partial^-_{\mathbb{X}}$ will map these states to $GC^-_{-3,-1}$ which is generated by more states, it is reasonable to assume that $\ker(\partial^-_{\mathbb{X}}) = 0$. As a consequence, then, $GH^-_{-1,0} = 0$ as well.

Now, these two minus flavors fit into the long exact sequence of Proposition 14 in the following way:

$$\cdots \longrightarrow \underbrace{GH^-_{0,0}}_{=0} \xrightarrow{f} GH^-_{-2,-1} \xrightarrow{\phi} \underbrace{\widehat{GH}_{-2,-1}}_{\cong\mathbb{F}} \xrightarrow{g} \underbrace{GH^-_{-1,0}}_{=0} \longrightarrow \dots. \tag{26}$$

Then, since this sequence is exact, $\ker(g) = \widehat{GH}_{-2,-1} = \mathrm{Im}(\phi)$, so ϕ is surjective, and since $\mathrm{Im}(f) = 0 = \ker(\phi)$, ϕ is moreover injective: as such, ϕ defines an isomorphism and we have $GH^-_{-2,-1} \cong \widehat{GH}_{-2,-1} \cong \mathbb{F}$, which proves this statement.

2. We shall strive to prove by induction. Observe at once that by 1, the base case of $k = 1$ is satisfied. Then, assume that for some m, $GH^-_{-2m,-m} \cong \mathbb{F}$. Now, we take a version of the long exact sequence of Proposition 14 with $d = -2m$ and $s = -m$:

$$\cdots \longrightarrow GH^-_{-2m,-m} \xrightarrow{U,f_1} GH^-_{-2(m+1),-(m+1)} \xrightarrow{f_2} \widehat{GH}_{-2(m+1),-(m+1)} \longrightarrow \ldots, \qquad (27)$$

but notice that $\widehat{GH}_{-2(m+1),-(m+1)} = 0$ for all $m \geq 1$. Thus, we must have that all of $GH^-_{-2(m+1),-(m+1)}$ is in the kernel of f_2, therefore in the image of f_1, and so that f_1 is actually surjective. Given the induction hypothesis, this leaves us only two options: either i) $GH^-_{-2(m+1),-(m+1)} = 0$, or ii) $GH^-_{-2(m+1),-(m+1)} \cong \mathbb{F}$.

Let us show that i) yields a contradiction. Indeed, if it were true, then we could devise a function $f_2' : 0 \to GH^-_{-2(m+1),-(m+1)}$, with the property that $(f_2 \circ f_2') = \mathrm{id}_0$. As such, by the splitting lemma, we would need to have that $GH^-_{-2m,-m} \oplus 0 = GH^-_{-2(m+1),-(m+1)} = 0$ – but this is an obvious contradiction! As such, we must reject i) and so say that ii) holds. That is, we must have $GH^-_{-2(m+1),-(m+1)} \cong \mathbb{F}$, which completes the induction step.

Since the base and induction steps have been completed, we conclude that $GH^-_{-2k,-k} \cong \mathbb{F}$ for all $k \in \mathbb{N}$.

3. Let us use the following version of the long exact sequence in Proposition 14:

$$\cdots \longrightarrow \widehat{GH}_{d,s} \longrightarrow GH^-_{d+1,s+1} \longrightarrow GH^-_{d-1,s} \longrightarrow \widehat{GH}_{d-1,s} \longrightarrow \ldots, \qquad (28)$$

together with the corollary that if $\widehat{GH}_{d+1,s} = \widehat{GH}_{d,s} = 0$, then $GH^-_{d+2,s+1} \cong GH^-_{d,s}$. Let's distinguish cases.

   (a) $GH^-_{d,s}$ s.t. $s > 1$: trivially, these are all zero.

   (b) $GH^-_{d,s}$ s.t. $s < -1$: by Propositions 12 and 13, we have that the corollary always holds for $s < -1$. As such, any $GH^-_{d,s}$ with $s < -1$ is congruent to one for which $s \geq -1$. Let δ be that Maslov grading so that $GH^-_{d,s} \cong GH^-_{\delta,-1}$. Then, there are three possibilities.

   - $\delta = -2$: in this case, $GH^-_{d,s} \cong GH^-_{-2,-1} \cong \mathbb{F}$.
   - $\delta = -3$: for this case, inspect a version of the long exact sequence:

$$\ldots \xrightarrow{f} \underbrace{GH^-_{-1,0}}_{=0} \xrightarrow{g} GH^-_{-3,-1} \xrightarrow{h} \underbrace{\widehat{GH}_{-3,-1}}_{=0} \longrightarrow \ldots. \qquad (29)$$

   So, we see that $GH^-_{-3,-1} = \ker(h) = \mathrm{Im}(g)$, so that g is surjective, and moreover that $\ker(g) = \mathrm{Im}(f) = 0$, so it is also injective. Thus, g defines an isomorphism, and as such $GH^-_{d,s} \cong GH^-_{-3,-1} \cong GH^-_{-1,0} = 0$.

- $\delta \neq -2$ and $\delta \neq -3$: then, $GH^-_{\delta,-1} \cong GH^-_{\delta+6,2} = 0$.

(c) $GH^-_{d,s}$ s.t. $(d,s) \in \{(-3,-1),(-2,-1),(-2,0),(-1,0),(-1,1),(0,1)\}$: all of these are known (by computations earlier in this proof), except for $(d,s) = (-2,0)$ and $(d,s) = (-1,1)$. It is left as an exercise to show that these are trivial.

(d) $GH^-_{d,s}$ otherwise: we can relate these unconcernedly with a $GH^-_{d',s>1} = 0$ via the corollary.

Collecting all that we have learned in this point and the previous two, we may conclude that 3. is now indeed shown true.

Since all points have been shown, the proposition is proven. □

As such, we have now effectively shown that

$$GH^-_{d,s} = \begin{cases} \mathbb{F} & \text{if} \quad (d,s) = (0,1) \quad \text{or} \quad (d,s) \in \{(-2k,-k)\}_{k=1}^{\infty} \\ 0 & \text{otherwise} \end{cases}, \tag{30}$$

completing Exercise 4.8.2(b) of the book (at long last).

3 Conclusion(s)

The computations that we performed to find the relevant tilde, hat, and minus-flavored homologies might seem random and arbitrary. And to some degree, they are. However, that does not make them completely useless. As was already hinted at last week, these homologies actually are knot invariants in some sense: $\widetilde{GH}(\mathbb{G})$ through Theorem 4.4.4, and $\widehat{GH}(\mathbb{G})$ and $GH^-(\mathbb{G})$ through Theorem 4.6.19.

In fact, we can even go one step further, by means of the following Theorem.

Theorem 16. *(4.7.6 of the book) For some knot K, we have*

$$\chi(\widehat{GH}(K)) = \Delta_K(t), \tag{31}$$

where $\Delta_K(t)$ is the symmetrized Alexander polynomial belonging to K, and χ is the graded Euler characteristic*:*

$$\chi(X) := \sum_{d,s} (-1)^d \cdot \dim(X_{d,s}) \cdot t^s, \tag{32}$$

for $X := \bigoplus_{d,s} X_{d,s}$.

Proof. Please see the book for the proof. □

It's rather cool that we can relate two seemingly very distant concepts – grid homologies and the Alexander polynomial – to each other in this way. It also opens up possible opportunities for calculating the Alexander polynomial in an alternate way than is standard. It is doubtful that for the cases we've seen so far it would save time and/or effort, but it is still possible.

Exercises

Below you can find some exercises that might tickle your fancy.

1. Prove that:

   (a) $\mathbb{F}[X]/(\mathbb{F}[X]X) \cong \mathbb{F}$;

   (b) $\mathbb{F}[X,Y]/(\mathbb{F}[X,Y](X+Y)) \cong \mathbb{F}[X]$.

   (These are final steps of Propositions 7 and 9.)

2. Show Proposition 13.3(d), i.e. that $\widehat{GH}_{-6,-3} = \widehat{GH}_{-4,-1} = 0$.

3. Show that $GH^-_{-2,0} = 0$ and $GH^-_{-1,1} = 0$, completing Proposition 14.3(c).

4. Use Theorem 16 to calculate the Alexander polynomials $\Delta_{\mathcal{O}}(t)$ and $\Delta_{T_{2,3}}(t)$ (and compare with literature).

5. Instead of the right-handed trefoil $T_{2,3}$, one could also consider the *left-handed* trefoil $T_{-2,3}$ (see Figure 3.3 on page 45 of the book). The same computations can be done.

   (a) Are (somewhat) similar homologies expected?

   (b) Modify the code at the end of these notes (or write your own, great fun!) to find the equivalent of Table 1 for $T_{-2,3}$.

   (c) Repeat the reasoning of Proposition 13 to come to your conclusions.

   (d) Use Theorem 16 to calculate the Alexander polynomial $\Delta_{T_{-2,3}}(t)$.

Listing 1: Code to obtain the information in Table 1.

```
import itertools as its
from collections import Counter

# Function to determine whether or not a pair is less than another one
def less(pair):
  a, b = pair
  if a[0] < b[0] and a[1] < b[1]:
    return True
  else:
    return False

# Function to calculate the Alexander grading s
def A(state):
  s = 0

  # Given the state, look up the winding number in grid and add to s
  for i in range(5):
    idx = state[1][i]
    s += grid[i][idx]

  # Calculate the final grade and return it
  grade = -1 * s - 5
  return grade

# Function to calculate the Maslov grading d
def M(state):
  J_xx, I_Ox, I_xO = 0, 0, 0
  Os = [(i + .5, 4.5 - i) for i in range(0, 5)]
  coords = list([(state[0][i], state[1][i]) for i in range(5)])

  # Comparisons between ordered pairs of coordinates
  comps = its.permutations(coords, 2)
  for pair in comps:
    if less(pair):
      J_xx += 1

  # Comparisons between the coordinates and Os
  comps2 = its.product(Os, coords)
  for pair in comps2:
    if less(pair):
      I_Ox += 1

  # Comparisons between the Os and coordinates.
  comps3 = its.product(coords, Os)
  for pair in comps3:
    if less(pair):
      I_xO += 1

  # Do some final calculation, and then return the grade
  J_xO = (I_xO + I_Ox) / 2
  grade = J_xx - 2 * J_xO + 1
```

```
  return int(grade)

# These are the winding numbers of the T_{2,3} grid; grid[x][y] <-> (x,y)
grid = [[0, 0, 0, 0, 0, 0], [0, 0, -1, -1, -1, 0], [0, -1, -2, -2, -1, 0], [0, -1, -2, -1, 0, 0], [0, -1, -1, 0, 0, 0], [0, 0, 0, 0, 0, 0]]

# Generate the possible permutations
base = (0, 1, 2, 3, 4)
perms = list(its.permutations(base))

# Create all the states
states = []
for i in perms:
  states.append([base, i])

# Calculate the Alexander and Maslov gradings for each state, and append them to each state
for i in range(len(states)):
  a = A(states[i])
  m = M(states[i])
  states[i] = [states[i], a, m]

# For a given Alexander grading s, create an array with the possible Maslov gradings
s = 1
maslovs = []
for i in states:
  if i[1] == s:
    maslovs.append(i[2])

print(Counter(maslovs))
```