Computations (Robbert Scholtens)

plan:
1. Definitions
2. Unknot
3. Trefoil $T_{2,3}$
4. Outro & Exercises

I Definition

def The simply blocked grid complex ("hat") is

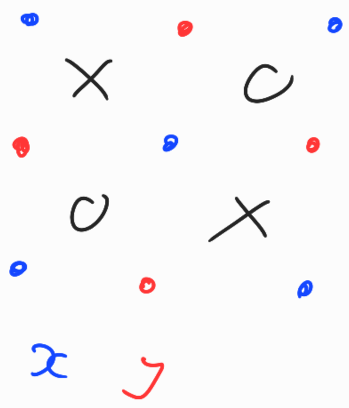
$$\widehat{GC}(\mathbb{G}) := GC^-(\mathbb{G})/V_i$$

The " " " homology:

$$\widehat{GH}(\mathbb{G}) := (\widehat{GC}(\mathbb{G}), \partial_\mathbb{X}^-)$$

notation $GH_{d,s} := GH_d(\mathbb{G}, s)$ For $\sim, \wedge, -$

II Unknot



lemma 1 We have

- $A(x) = M(x) = -1$
- $A(y) = M(y) = 0$

lemma 2 $\partial_\mathbb{X}^-$ acts on x and y as

$$\partial_\mathbb{X}^-(x) = (V_1 + V_2)y \qquad \partial_\mathbb{X}^-(y) = 0$$

prop 3 For the unknot O, $GH^-(O) \cong \mathbb{F}[U]$.

proof By Lemma 1, $GC^-_{-1,-1}$ is generated by x and $GC^-_{0,0}$ is gen. by y, both over $\mathbb{F}[V_1, V_2]$. By lemma 2, we conclude that the kernel of ∂_x^- is $\mathbb{F}[V_1, V_2]y$, and the image is $\mathbb{F}[V_1, V_2](V_1+V_2)y$. Thus,

$$GH^-_{-1,-1} = \frac{\ker(\partial_x^-) \cap GC^-_{-1,-1}}{\operatorname{im}(\partial_x^-) \cap GC^-_{-1,-1}} = 0.$$

So actually $GH^-_{0,0}$ is $GH^-(O)$, and it is

$$GH^-(O) = \frac{\ker(\partial_x^-)}{\operatorname{im}(\partial_x^-)} = \frac{\mathbb{F}[V_1, V_2]y}{\mathbb{F}[V_1, V_2](V_1+V_2)y} \cong \frac{\mathbb{F}[V_1, V_2]}{\mathbb{F}[V_1, V_2](V_1+V_2)}$$

$$\cong \mathbb{F}[U]$$ exercise! ∎

prop 4 $\widehat{GH}(O) \cong \mathbb{F}$.

proof To form $\widehat{GC}_{d,s}$, set $V_2 \equiv 0$. Our ∂_x^- acts as

$$\partial_x^-(x) = V_1 y \qquad \partial_x^-(y) = 0$$

So then, $\ker(\partial_x^-) = \mathbb{F}[V_1]y$ and $\operatorname{im}(\partial_x^-) = \mathbb{F}[V_1]V_1 y$.

$$\widehat{GH}(O) = \frac{\ker(\partial_x^-)}{\operatorname{im}(\partial_x^-)} = \frac{\mathbb{F}[V_1]y}{\mathbb{F}[V_1]V_1 y} \cong \mathbb{F}$$ exercise! ∎

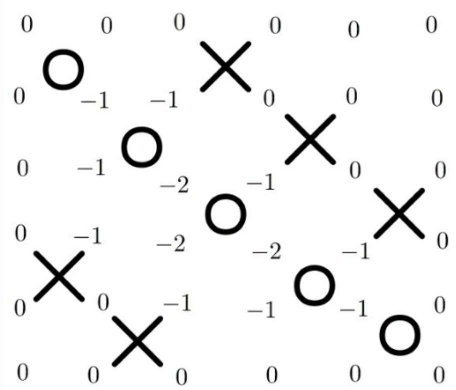
III Trefoil knot ($T_{2,3}$)

We need a bit more machinery.

prop 5 (4.6.15) We have

$$\widetilde{GH}_{d,s} \cong \widehat{GH}_{d,s} \oplus \widehat{GH}_{d+1,s+1}^{\oplus 4} \oplus \widehat{GH}_{d+2,s+2}^{\oplus 6} \oplus \widehat{GH}_{d+3,s+3}^{\oplus 4} \oplus \widehat{GH}_{d+4,s+4}$$

$n=5$



prop 6 (4.6.10) There is a long exact sequence

$$\cdots \to GH^-_{d+2,s+1} \to GH^-_{d,s} \xrightarrow{f} \widehat{GH}_{d,s} \xrightarrow{g} GH^-_{d+1,s+1} \to \cdots$$

↳ $\ker(g) = \mathrm{im}(f)$

Lemma

1. There are no states on $T_{2,3}$ w/ $s>1$.
2. There is 1 state w/ $s=1$.

proof

1. $A(x) = A'(x) - 5$; $A'(x) := -\sum_{p \in x} w(p)$

↳ prop 4.7.2 $\qquad \underline{\leq 6}$

2. Evident? ∎

prop 7 $\widehat{GH}_{0,1} \cong \mathbb{F}$.

proof $\widetilde{GC}_{0,1} = \mathbb{F}a \cong \mathbb{F}$. But also

- $\ker(\tilde{\partial}_{0,x}) \cap \widetilde{GC}_{0,1} \cong \mathbb{F}$
- $\mathrm{im}(\tilde{\partial}_{0,x}) \cap \widetilde{GC}_{0,1} = 0$

Therefore, $\widetilde{GH}_{0,1} \cong \mathbb{F}$. Then, by Proposition 5, we get

$$\widetilde{GH}_{0,1} \cong \widehat{GH}_{0,1} \oplus \cancel{\cdots} \to 0 \quad \Rightarrow \quad \widehat{GH}_{0,1} \cong \mathbb{F}.$$

The problems:

- How many states w/ given d, s?
- The kernels might not be trivial.

$d \backslash s$	-5	-4	-3	-2	-1	0	1
-6	1	0	1	0	0	0	0
-5	0	5	5	0	0	0	0
-4	0	0	20	6	1	0	0
-3	0	0	5	30	5	0	0
-2	0	0	0	10	20	0	0
-1	0	0	0	0	5	5	0
0	0	0	0	0	0	0	1

prop 6

1. $\widehat{GH}_{-1,0} \cong \mathbb{F}$
2. $\widehat{GH}_{-2,-1} \cong \mathbb{F}$
3. $\widehat{GH}_{d,s} = 0$ for all other combos.

proof

1. By the table, there 5 states generating $\widetilde{GC}_{-1,0}$, and so $\widetilde{GH}_{-1,0} = \mathbb{F}(b_1 \cdots b_5) \cong \mathbb{F}^{\oplus 5}$. Then use prop.

5:

$$\widetilde{GH}_{-1,0} \cong \mathbb{F}^{\oplus 5} \cong \widehat{GH}_{-1,0} \oplus \underbrace{\widehat{GH}_{0,1}^{\oplus 5}}_{\cong \mathbb{F}^{\oplus 5}} \oplus \cancel{\cdots} \to 0$$

$$\Rightarrow \widehat{GH}_{-1,0} \cong \mathbb{F}.$$

2. Deduce that $\widetilde{GH}_{-6,-5} \cong \mathbb{F}$ and $\widetilde{GH}_{-5,-4} \cong \mathbb{F}^{\oplus 5}$.

But they also feature in prop. 5:

$$\mathbb{F} \cong \widetilde{GH}_{-6,-5} \cong \cancel{\widehat{GH}_{-6,-5}} \oplus \cancel{\widehat{GH}_{-5,-4}^{\oplus 5}} \oplus \cancel{\widehat{GH}_{-4,-3}^{\oplus 6}} \oplus \cancel{\widehat{GH}_{-3,-2}^{\oplus 5}} \oplus \widehat{GH}_{-2,-1}$$

$$\mathbb{F}^{\oplus 5} \cong \widetilde{GH}_{-5,-4} \cong \cancel{\widehat{GH}_{-5,-4}} \oplus \cancel{\cdots} \oplus \widehat{GH}_{-2,-1}^{\oplus 5} \oplus \underbrace{\widehat{GH}_{-1,0}}_{\cong \mathbb{F}}.$$

Now assume $\widehat{GH}_{d,s} \cong \mathbb{F}^{\oplus k}$ $k \in \mathbb{N} \cup \{0\}$, then conclude that $\widehat{GH}_{-2,-1} \cong \mathbb{F}$.

3. ∘ By 1, $\widehat{GH}_{-6,-5} = \dots = \widehat{GH}_{-3,-2} = 0$

∘ $\widetilde{GH}_{-5,-5} = 0$. Thus,

$$\widetilde{GH}_{-5,-5} = 0 \cong \cancel{\widehat{GH}_{-5,-5} \oplus \dots \oplus \widehat{GH}_{-1,-1}}$$

∘ $\widetilde{GH}_{-6,-4} = 0$. Thus,

$$\widetilde{GH}_{-6,-4} = 0 \cong \cancel{\widehat{GH}_{-6,-5}} \oplus \cancel{\dots} \oplus \cancel{\widehat{GH}_{-2,0}}$$

∘ the rest is an exercise! ∎

Summarizing,

$$\widehat{GH}_{d,s} = \begin{cases} \mathbb{F} & (d,s) = \{(0,1), (-1,0), (-2,-1)\} \\ 0 & \text{otherwise} \end{cases},$$

which is exercise 4.8.2(a) of the book.

theorem (4.7.6) For some knot K,

$$\chi(\widehat{GH}(K)) = \Delta_K(t),$$

where χ is given by

$$\chi(X) := \sum_{d,s} (-1)^d \cdot \dim(X_{d,s}) \cdot t^s,$$

where $X := \bigoplus_{d,s} X_{d,s}$.

proof Book. ∎

- $\mathbb{F}[X] / \mathbb{F}[X]X \cong \mathbb{F}$ $\rightsquigarrow$ $p = p_0 + Xp_1 \equiv p_0$

- $\mathbb{F}[X,Y] / \mathbb{F}[X,Y](X+Y) \cong \mathbb{F}[X]$.

  $\rightsquigarrow$ replace Y by X