TOPICS IN TOPOLOGY, PRESENTATION 01-02-2021: ALEXANDER POLYNOMIAL VIA SEIFERT SURFACES

A. LOS

Contents

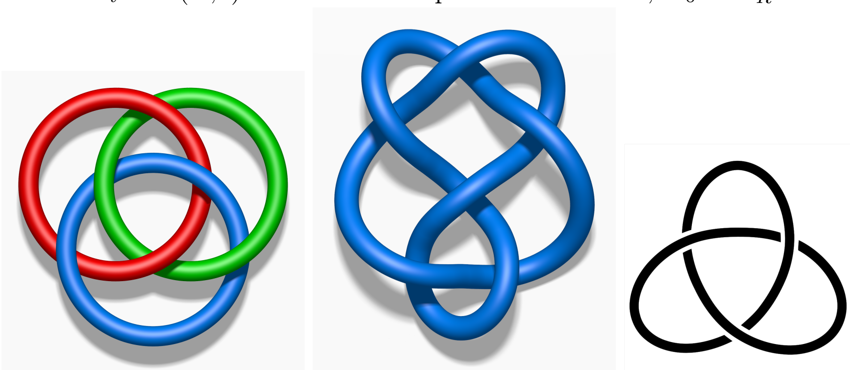
From **Grid Homology for Knots and Links**, by Ozsvath, Stipsicz and Szabo [PSO15], we cover today parts of sections 2.2 (Seifert surfaces), 2.3 (Signature and unknotting number), 2.4 (The Alexander polynomial). For this definition of the Alexander polynomial of a link/knot we need the Seifert matrix and for that we need Seifert surfaces. (Our main purpose is to give intuition and ideas. Precise definitions and proofs can be read in the book.)

Introduction to knot theory

Defintion: A **n-Link** is an embedding of n circles into three-dimensional space. Knots are 1-links, with one component.

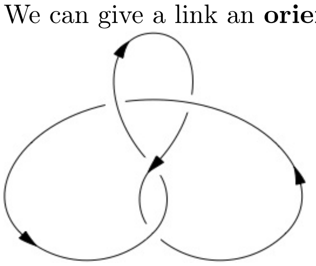
Definition: We define an equavalence on the set of embeddings by regarding two links equivalent iff they are ambient isotopic. (I.e. if they can be continuously deformed into eachother.) Or more precise:

Two embeddings $f, g : \coprod_n S^1 \hookrightarrow \mathbb{R}^3$ are ambient isotopic if theres is an isotopy $H : \mathbb{R}^3 \times I \to \mathbb{R}^3$ such that $H_t = H(-, t)$ is an homeomorphism for all $t \in I$, $H_0 = id_{R^3}$ and $H_1 \circ f = g$.



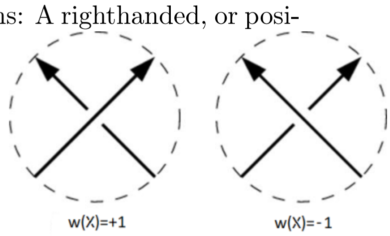
The borromean rings link, the knot 6_2, the trefoil knot.

We can give a link an **orientation**, by choosing directions of the lines.



An example of the figure-eight knot.

Defintion: We can distinct two different oriented crossings in diagrams: A righthanded, or positive, crossing and a lefthanded, or negative, crossing, see the next picture.



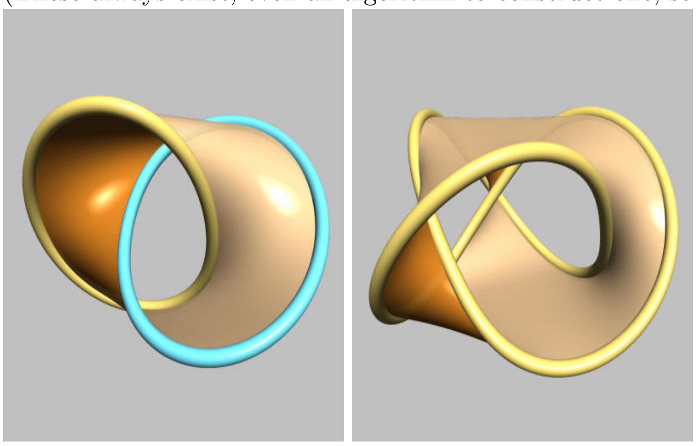
Definition: A **link invariant** is a property associated to a link (often a number or polynomial) that is invariant under ambient isotopies of the link. (To be able to say things about knots/links or to distinguish some link from others, we want to get some information from the link, to say which properties it has. We want properties that do not change under ambient isotopy (changing the link a little bit continuously). We'll look at one of those properties.)

Definition (2.1.12): For oriented (2-)links the **linking number** is number of 'windings' of one component around the other: We look at all the crossings where one component crosses with another component. The linking number is half the sum of the signs of these crossings.

2.2 Seifert surface

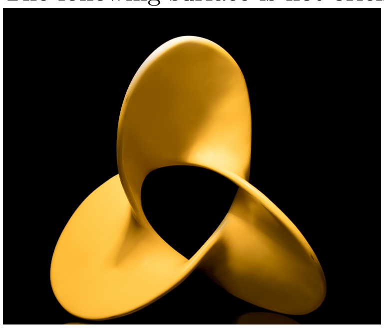
Definition 2.2.1: A smoothly embedded, compact, connected, oriented surface-with-boundary Σ in $\mathbb{R}^3$ is a **Seifert surface** of the oriented link $\overrightarrow{L}$ if $\partial\Sigma = L$, and the orientation induced on $\partial\Sigma$ agrees with the orientation specified by $\overrightarrow{L}$.

(These always exist, even an algorithm to construct one, see Appendix B.3 on page 382.)



Seifert surfaces for the Hopf-link and the trefoil.

The following surface is not orientable, so *not* a Seifert surface for the trefoil knot.



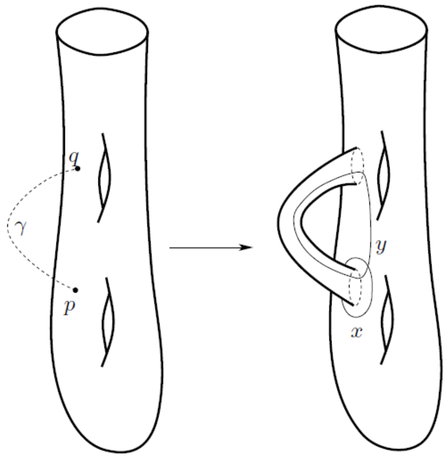
Definition: The **genus** of a closed orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected. It is equal to the number of handles on it. (E.g. a ball has genus zero and a torus genus 1.)

Definition 2.2.4: The Seifert **genus** $g(\overrightarrow{L})$ is the minimal genus of any Seifert surface for $\overrightarrow{L}$, where we obtain a closed surface by glueing in a disk along each boundary component. This genus is useful, but can be difficult to obtain from a picture. The easiest way to compute it is using the Euler characteristic:

Lemma: The genus of a connected compact orientable surface Σ is related to the Euler characteristic χ by the formula $\chi(\Sigma) = 2 - 2g(\Sigma)$ (or $\chi(\Sigma) = 2 - 2g(\Sigma) - b(\Sigma)$ if we allow Σ to have b boundary components).

Example: For the trefoil example above we can see that the euler characteristic is -1, so $-1 = 2 - 2g(\Sigma) - 1$, thus $g = 1$. (This is in fact the Seifert genus of the trefoil, because there is no Seifert surface for the trefoil with genus zero.)

Definition/lemma: Stabilisation of a Seifert surface leads to a different Seifert surface for the same link. Connect two interior point p, q of the surface by an arc γ outside the surface that approaches p and q from the same side of the surface. Delete small neighbourhoods of p and q



from the surface and add an annulus around γ:

Remark 2.2.6: The **linking number** $lk(\overrightarrow{K}_1, \overrightarrow{K}_2)$ is the algebraic intersection number of a Seifert surface for $\overrightarrow{K}_1$ with the oriented knot $\overrightarrow{K}_2$. See the example of the Hopf link below, where $lk = 1$. But we will need the linking number of two loops on the seifert surface, which is in general slightly more difficult to observe.

2.3 Seifert matrix

Seifert form represented by a Seifert Matrix.

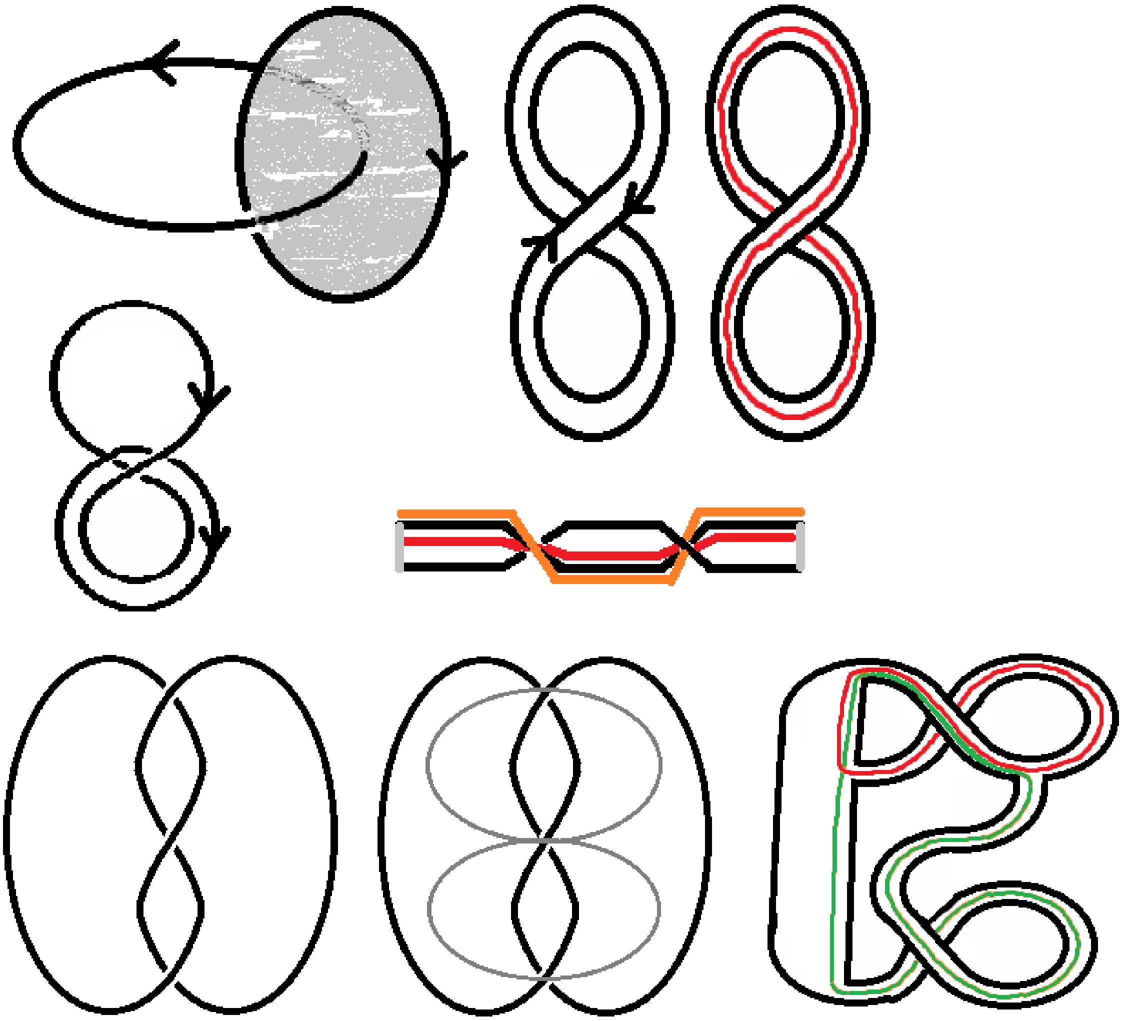
Definition: For a connected surface Σ the **first homology group** with coefficients in $\mathbb{Z}$, $H_1(\Sigma, \mathbb{Z})$ is the group of loops on the surface at a basepoint modulo homotopy and conjugation. A basis for this free abelian group (i.e. Z-module) of closed loops on the surface through a basepoint can be represented by specific embedded circles that are not related by homotopy or conjugacy. (For example a torus has two classes of such independent loops, so has first homology group $\mathbb{Z} \oplus \mathbb{Z}$.)

Definition 2.3.1: The **Seifert form** S for the Seifert surface Σ of the link $\overrightarrow{L}$ is a $\mathbb{Z}$-bilinear map $S : H_1(\Sigma, \mathbb{Z}) \times H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}$. It is defined for $x, y \in H_1(\Sigma; \mathbb{Z})$ by $S(x, y) = lk(\gamma_x, \gamma_y^+)$, where γ_x is a loop representing the class x and where the γ^+ indicates that we lift the loop γ a bit to the positive side of the oriented surface.

Lemma: $H_1(\Sigma, \mathbb{Z}) \cong \bigoplus_{2g+n-1} \mathbb{Z}$ is generated by the $2g+n-1$ generators $\sigma_1, \ldots, \sigma_g, \gamma_1, \ldots, \gamma_g, \beta_1, \ldots, \beta_{n-1}$, where n is the number of component of the link, which is the number of boundary components of the surface. g is the genus of the surface where we glued in n disks. **Proofsketch:** For each torus-like hole there is a loop γ through the hole and a loop σ around the hole. For each boundary component after the first there is a loop β around it.

Definition: The form is represented by a **Seifert matrix** $S(i, j) = (lk(\alpha_i, \alpha_j^+))$, the $(2g + n - 1) \times (2g + n - 1)$ square matrix of their pairwise linking numbers. (One loop lifted a bit from the surface.) This is a bilinear form. (E.g. walking a loop/circle twice, doubles the linking number.)

Signature and Alexander polynomial, have analogues in grid homology: τ-invariant and Poincaré polynomial.



Seifert matrix Hopf link: (1). Seifert matrix trefoil: $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

2.4 Alexander polynomial

Definition: The **Alexander polynomial** $\Delta_{\overrightarrow{L}}(t)$ of an oriented link L with Seifert matrix S is given by $\Delta_{\overrightarrow{L}}(t)$ =det$(t^{-1/2}S - t^{1/2}S^T)$.

Theorem 2.4.1: $\Delta_{\overrightarrow{L}}(t)$ is independent from choices of Seifert surface and Seifert matrix, so is an invariant of the link.

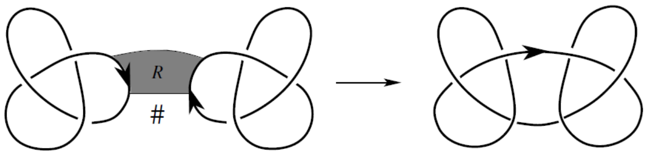
Proof: By Theorem 2.2.2, any two Seifert surfaces for a link can be made into ambient isotopic Seifert surfaces by use of stabilisation. By Lemma 2.3.4, for a Seifert surface Σ with matrix S, a sabilisation Σ' of Σ has a matrix S' of the form $\begin{pmatrix} S & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} S & 0 & 0 \\ \xi^T & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ for some vector ξ. (Where the 1 comes from $lk(x, y)$ for the loops x and y, see the picture of stabilisation. The ξ comes from the linking of y with other generating loops.) These matrices and S lead to the same Alexander polynomial: $\Delta_{\overrightarrow{L}}(t) = \det(t^{-1/2}S - t^{1/2}S^T)$, and $\Delta_{\overrightarrow{L}}(t) = \det(t^{-1/2}S' - t^{1/2}S'^T) = \det\begin{pmatrix} t^{-1/2}S - t^{1/2}S^T & t^{-1/2}\xi & 0 \\ -t^{1/2}\xi^T & 0 & t^{-1/2} \\ 0 & -t^{1/2} & 0 \end{pmatrix}$ which is (expanding by the third row) $t^{1/2} \cdot \det\begin{pmatrix} t^{-1/2}S - t^{1/2}S^T & 0 \\ -t^{1/2}\xi^T & t^{-1/2} \end{pmatrix}$. Expanding by the second column gives $\det(t^{-1/2}S - t^{1/2}S^T)$. The same result holds for the transpose case.

Lemma 2.4.3: For a knot K the Alexander polynomial is symmetric: $\Delta_K(t) = \Delta_K(t^{-1})$.
Proof: Exercise 2 below.

Lemma (Exc. 2.4.4 (a)): For a knot the Alexander polynomial is in $\mathbb{Z}[t, t^{-1}]$. **Proof:** For a knot K with Seifert surface Σ and genus g, $H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, so the Seifert matrix S is $2g \times 2g$. Each term in the determinant is the product of an even number of factors, each of which is zero or $\pm t^{1/2}$ or $\pm t^{-1/2}$ or a sum of those. So each factor t is raised to the power $z \cdot 2 \cdot 1/2$ for some $z \in \mathbb{Z}$.

Definition: For knots the **degree** $d(K)$ of $\Delta_K(t) = a_0 + \sum_{i=1}^{n} a_i(t^i + t^{-i})$ is the maximal d for which $a_d \neq 0$.

Definition: The **connect-sum** $K_1 \# K_2$ of two knots K_1 and K_2 is the knot that is formed by cutting both knots open and connecting the ends K_1 to the ends of K_2 corresponding with the orientations, as in the picture.



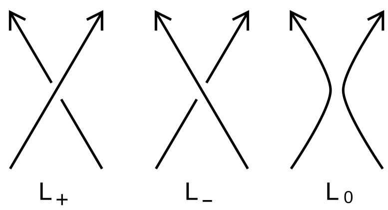
Theorem 2.4.6: Suppose that the knot K has Alexander polynomial $\Delta_K(t)$ of degree $d(K)$. Then
(1) The Seifert genus $g(K)$ of K satisfies $g(K) \geq d(K)$.
(2) For any two knots K_1 and K_2, $\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$.
(3) For any knot K, $\Delta_K(1) = 1$.

Proof: (1) The Seifert surface Σ that has genus $g(K)$ has a $2g \times 2g$ matrix, so cannot have higher degree than $d(t^{\pm 1/2 \cdot 2g}) = g$.
(2) The connect-sum of knots induces a connecting of two Seifert surfaces along part of their boundaries to a new Seifert surface for the connect-sum. The Seifert matrix is $\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$ and leads to the product of the two determinants.
(3) Choose a basis $\{\alpha_i, \beta_j\}_{i,j=1}^{g}$ for $H_1(\Sigma)$ so that $\#(\alpha_i \cap \beta_i) = 1$ and all other pairs of curves are disjoint. Then for the Seifert matrix S, the matrix $S^T - S$ decomposes as blocks of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which have determinant 1.

There is an alternative (easy) way of computing the Alexander polynomial via skein relations. (And there are more ways to calculate it, e.g. via a matrix for the crossings and adjacent regions.) Many link polynomials use **Skein relations**, where you can replace a link which at some point locally looks like one of the three following local pictures, by two links with the other two pictures instead (where we count those two links with some factor). In this way you can replace your link by simpler links until you end with only trivial unknots, all with some factor.

Lemma: The skein relations for the Alexander polynomial are the following:
$\Delta(O) = 0$, for O the unknot.

$\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta(L_0)$.



Exercises

In the book there are a lot of good exercises. Some exercises we present below. To check your computations of the Alexander polynomial: Look up the link in the knot atlas (katlas.org).

Exercise 1: Compute the Alexander polynomial for the trefoil knot. (See below for the solution.)

Exercise 2: Proof Lemma 2.4.3: For a knot K the Alexander polynomial is symmetric: $\Delta_K(t) = \Delta_K(t^{-1})$. (See below for the solution.)

Exercise 3: Exercise 2.2.5: Show that the unique knot with $g(K) = 0$ is the unknot.

Exercise 4: For the figure eight knot, find a Seifert surface, its matrix and its Alexander polynomial.

References

[PSO15] Zoltan Szabo Peter S. Ozsvath, Andras I. Stipsicz. *Grid Homology for Knots and Links.* Mathematical surveys and monographs. American Mathematical Society, Providence, Rhode Island, 208 edition, 2015.

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Solution to Exercise 1: To compute the Alexander polynomial for the trefoil knot. First recall that the Seifert matrix S is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. The results in the Alexander poloynomial: $\Delta_{3_1}(t)$ =det$(t^{-1/2}S-t^{1/2}S^T)$ = det$(\begin{pmatrix} t^{-1/2} & -t^{-1/2} \\ 0 & t^{-1/2} \end{pmatrix} - \begin{pmatrix} t^{1/2} & 0 \\ -t^{1/2} & t^{1/2} \end{pmatrix})$ =det$\begin{pmatrix} t^{-1/2}-t^{1/2} & -t^{-1/2} \\ -t^{1/2} & t^{-1/2}-t^{1/2} \end{pmatrix}$. This is $(t^{-1/2}-t^{-1/2})^2 - (-t^{-1/2}\cdot t^{1/2}) = t^{-1} - 2 + t - (-1)$, so we conclude $\Delta_{3_1}(t) = t - 1 + t^{-1}$. We check that this is indeed in $\mathbb{Z}[t, t^{-1}]$ and indeed $\Delta(1) = 1 + 1 - 1 = 1$.

Solution to Exercise 2: For a knot K with Seifert surface Σ and genus g, $H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, so the Seifert matrix S is $2g \times 2g$. Then $\Delta_K(t^{-1}) = (-1)^{2g}\text{det}(t^{-1/2}S^T - t^{1/2}S) = \Delta_K(t)$.