A genus bound for grid homology

Topics in Topology Lecture 7

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1 Theory

In this hand-out, we will try to solve Proposition 7.2.2, which states that the simply blocked grid homology is bounded from above (with respect to the Alexander degree) by the genus of the knot. We will prove this statement by creating a Seifert surface out of (a variation on) the winding matrix of a grid diagram. This surface then induces an invariant which only depends on the grid diagram, and which is at most the maximum value of the Alexander function from Roelien's talk, thus giving an upper bound to the simply blocked grid homology. Moreover, we will show that this invariant, which in fact is the genus of the Seifert surface, is at most the genus of the involved knot, and we show that there is a grid diagram which attains this maximum. In this hand-out, we will also use a running example of the left-handed trefoil knot to clarify the used process.

1.1 Minimal matrices

We denote the set of $(n \times n)$-matrices with integral coefficients by $M_n(\mathbb{Z})$ (and call them **integral matrices**) and with non-negative integral coefficients by $M_n(\mathbb{N})$. Moreover, for $1 \leq k \leq n$, define $C^k \in M_n(\mathbb{N})$ by $(C^k_{ij}) = \delta^k_j$ and $R^k \in M_n(\mathbb{N})$ by $(R^k_{ij}) = \delta^k_i$, that is, C_k is the matrix with ones in the k-column and zero anywhere else, and R_k is the matrix with ones in the k-th row and zero anywhere else. We can use those definitions to find a relation between different integral matrices.

Definition 1.1 *We define the relation $\sim$ on $M_n(\mathbb{Z})$ by*

$$A \sim B \text{ if and only if } A - B = \textstyle\sum_{k=1}^n (s_k R_k + t_k C_k) \text{ for } s_k, t_k \in \mathbb{Z}.$$

This relation can be phrased in words by saying that two matrices are related if and only if we can get from one matrix to the other by adding (and integer amount of) rows or columns with only ones. We leave it as an exercise to prove that this relation is an equivalence relation.
We now want to find the 'simplest' integral matrix relating to a given integral matrix, that is, a suitable representative for each equivalence class. To accomplish this, we define such a matrix, which we call a minimal matrix.

Definition 1.2 *The **complexity** of $A \in M_n(\mathbb{Z})$ is $c(A) := \sum_{i=1}^n \sum_{j=1}^n A_{i,j}$. A matrix $A \in M_n(\mathbb{N})$ is **minimal** if and only if for all $B \in M_n(\mathbb{N})$ such that $A \sim B$, we have that $c(B) \geq c(A)$.*

A small proposition states that we can find a minimal matrix for every integral matrix.

Proposition 1.3 *For every integral matrix $C \in M_n(\mathbb{Z})$, there exists a minimal matrix $A \in M_n(\mathbb{N})$ such that $C \sim A$.*

The proof of this proposition is left as an exercise.
It might not be easy to see, at a first glance, if a matrix $A \in M_n(\mathbb{N})$ is minimal or not. There is, however, a simple criterion to find out if A is minimal.

Proposition 1.4 *Let $A \in M_n(\mathbb{N})$. Then A is minimal if and only if there exists a permutation $\sigma \in S_n$ such that $A_{k,\sigma(k)} = 0$ for all $1 \leq k \leq n$.*

Proof. Let $A \in M_n(\mathbb{N})$ and assume that there exists a permutation $\sigma \in S_n$ such that $A_{k,\sigma(k)} = 0$ for all $1 \leq k \leq n$. Now let $A' \in M_n(\mathbb{N})$ such that $A' \sim A$. Then there exists $s_k, t_k \in \mathbb{Z}$ $(1 \leq k \leq n)$ such that $A' - A = \sum_{k=1}^{n}(s_k R_k + t_k C_k)$. Rewriting this gives us that

$$A' = A + \sum_{k=1}^{n}(s_k R_k + t_k C_k). \tag{1}$$

We then find that

$$0 \leq (A')_{k,\sigma(k)} = A_{k,\sigma(k)} + s_k + t_{\sigma(k)} = s_k + t_{\sigma(k)} \tag{2}$$

for all $1 \leq k \leq n$, which in its turn gives us that

$$\begin{aligned} c(A') &= c(A + \sum_{k=1}^{n}(s_k R_k + t_k C_k)) = c(A) + c(\sum_{k=1}^{n}(s_k R_k + t_k C_k)) = c(A) + \sum_{k=1}^{n}(s_k + t_k) \\ &= c(A) + \sum_{k=1}^{n}(s_k + t_{\sigma(k)}) \geq c(A), \end{aligned} \tag{3}$$

which proves that A is minimal.
For the other implication, assume that there does not exist a permutation $\sigma \in S_n$ such that $A_{k,\sigma(k)} = 0$ for all $1 \leq k \leq n$. This then gives us that there is a row or column of A which consists solely of positive entries. We can thus substract some R_k or C_k such that the new matrix is still in $M_n(\mathbb{N})$, but with a lower complexity of $c(A) - n \leq c(A)$, which shows that A is not minimal.
Combining both implications yields the desired result. □

Now let $\mathbb{G}$ be a grid diagram of a knot K, and let H be a minimal matrix of $W(\mathbb{G})$ (which exists by Proposition 1.3). We will study this matrix H and its properties for a large part of the hand-out. One of those statements is that adjacent entries of H can only differ by one. First, we start on finding H in our example, the left-handed trefoil.

Example 1.5 *Let $\mathbb{G}$ be given by Figure 1.*

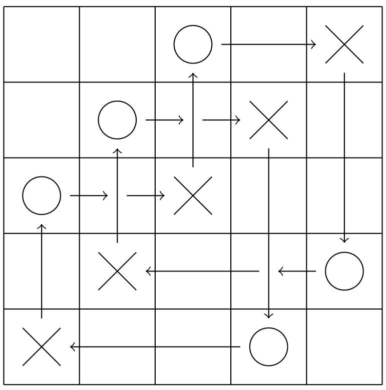

Figure 1: A grid diagram for the left handed trefoil knot.

Then the winding matrix $W(\mathbb{G})$ is given by

$$W(\mathbb{G}) := \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

which is already minimal.

Now we will find some properties of this matrix H in general.

Proposition 1.6 *Let H as above. Then for $1 \leq i, j \leq n$, we have that*

$$|H_{i,j} - H_{i,j+1} - H_{i+1,j} + H_{i+1,j+1}| \leq 1, \tag{5}$$

where addition is taken modulo n.

Proof. In our winding matrix, consider a square of adjacent corners and denote them by a, b, c and d. We want to find $w(a) - w(b) - w(c) + w(d)$, and we can do this by considering the tangle inside the square with corners a, b, c, d. For such tangles, we then have that there are four possibilities up to rotation for which there is not an X or O in the centre, namely those from Figure 2.

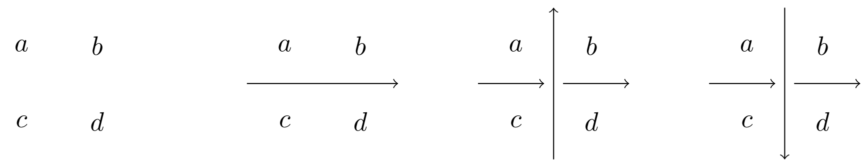

Figure 2: Four possibilities for tangles between four corners.

In the first situation, we have that $w(a) = w(b) = w(c) = w(d)$, and thus that $w(a) - w(b) - w(c) + w(d) = 0$. In all the other situations, we have that $w(a) = w(c) + 1$ and $w(b) = w(d) + 1$, for which we find that $w(a) - w(b) - w(c) + w(d) = w(c) + 1 - (w(d) + 1) - w(c) + w(d) = 0$. Then, there are also two possibilities up to rotation for which there is an X or O in the centre, namely those from Figure 3.

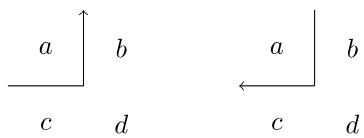

Figure 3: Two possibilities for tangles between four corners.

In the first situation, we have that $w(a) = w(b) + 1$ and that $w(b) = w(c) = w(d)$, and thus that $w(a) - w(b) - w(c) + w(d) = 1$. In the second situation, we have that $w(a) = w(b) - 1$ and that $w(b) = w(c) = w(d)$, for which we find that $w(a) - w(b) - w(c) + w(d) = -1$.
We thus find that

$$|a - b - c + d| \leq 1. \tag{6}$$

Now note that the sum $a-b-c+d$ does not change when we pass from the winding matrix $W(\mathbb{G})$ to any related matrix. As H is related to $W(\mathbb{G})$ by definition, we find that proposition must hold. $\square$

We can even up this result to find that adjacent entries of H differ by 1 at most.

Proposition 1.7 *Let H as above. Then for $1 \leq i, j \leq n$, we have that*

$$|H_{i,j} - H_{i,j+1}| \leq 1 \tag{7}$$

and that

$$|H_{i,j} - H_{i+1,j}| \leq 1, \tag{8}$$

where addition is taken modulo n.

We leave the proof of this proposition as an exercise.

1.2 A Seifert surface

From the last subsection, we found a minimal matrix H for every winding matrix of grid diagram $W(\mathbb{G})$, and we found that adjacent entries of H differ by at most 1. We will now use this minimal matrix to construct a Seifert surface for the knot that is represented by our grid diagram $\mathbb{G}$.

Construct the surface F_H as follows. First, for every $1 \leq i, j \leq n$, create $H_{i,j}$ squares, and call such a square $s^k_{i,j}$. (If $H_{i,j} = 0$, do not create any squares.) Then glue by the following rules:

- Glue the right edge of $s^k_{i,j}$ to the left edge of $s^k_{i,j+1}$ for $1 \leq k \leq \min(H_{i,j}, H_{i,j+1})$.
- Glue the bottom edge $s^{H_{i,j}-k}_{i,j}$ to the top edge of $s^{H_{i+1,j}-k}_{i+1,j}$ for $0 \leq k \leq \min(H_{i,j}, H_{i,j+1}) - 1$.

Again, addition is taken modulo n. In words, this means that we should glue horizontal lines from bottom to top and vertical lines from top to bottom. In Figure 4, the surface F_H corresponding to the H from Example 1.5 can be seen.

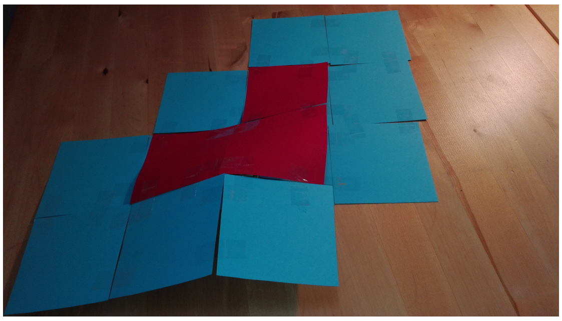

Figure 4: A picture of the surface F_H for H as in Example 1.5.

Note that it is clear from the picture that Figure 4 is a Seifert surface of the trefoil knot. It is actually true in general that the resulting surface is in fact a Seifert surface of our knot K.

Theorem 1.8 *Let K be knot, $\mathbb{G}$ a grid diagram of K and $W(\mathbb{G})$ its winding matrix. If H is a minimal matrix related to $W(\mathbb{G})$, then F_H is a Seifert surface.*

This theorem is left as an exercise to the reader.

One of the most elementary and interesting invariants of a 2-dimensional surface (or more general)is its genus.

Definition 1.9 *Let Σ be a 2-dimensional CW-complex (or a 2-dimensional surface). Then the **genus** of Σ is given by*

$$g(\Sigma) := 1 - \frac{1}{2}\chi(\Sigma) - \frac{1}{2}b(\Sigma), \tag{9}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ, and $b(\Sigma)$ its number of boundary components.
*The **Seifert genus** of a knot K is the minimal value of the genus of any Seifert surface of K.*

In this next part of this subsection, we will calculate the genus of our surface F_H. We will later see that this invariant is related to the maximal value of the Alexander degree, which will then impose an upper bound on the simply blocked grid homology.

To calculate the genus of F_H, we will need one more property of this surface, which we will prove in the following lemma.

Lemma 1.10 *Let H be a minimal matrix related to the winding matrix $W(\mathbb{G})$ of a grid diagram $\mathbb{G}$. If $x \in int(F_H)$ is a corner point of a square $s^k_{i,j}$, then x is a corner of exactly four squares $s^k_{i,j}$.*

Proof. This proof depends on the different types of corner points. By Proposition 1.6 and by Proposition 1.7, we find that all corner points can be divided into five different categories (up to rotations of 90 degrees), being

$$\begin{array}{c|c} a & a \\ \hline a & a \end{array}, \quad \begin{array}{c|c} a & a+1 \\ \hline a+1 & a+2 \end{array}, \text{ and } \begin{array}{c|c} a & a+1 \\ \hline a & a+1 \end{array} \quad (a \in \mathbb{N})$$

if there is no O or X at the centre of the four involved corners, and

$$\begin{array}{c|c} a & a \\ \hline a & a+1 \end{array}, \text{ and } \begin{array}{c|c} a & a+1 \\ \hline a+1 & a+1 \end{array} \quad (a \in \mathbb{N})$$

if there is an O or X at the centre of the four involved corners. By close inspection of those five cases, we can find that the statement is true. □

We are now ready to find the genus of F_H.

Theorem 1.11 *The genus of F_H is given by*

$$g(F_H) = \frac{1}{2}\theta(H) + \frac{n-1}{2}, \tag{10}$$

where $\theta(H)$ is the sum of the averages of the corners around each O and X.

Proof. First, we will compute the Euler characteristic of F_H. To find this, we note that

$$\begin{aligned} 0 &= \#\{\text{squares in } F_H\} - \frac{1}{2}\#\{\text{edges of squares in } F_H\} + \frac{1}{4}\#\{\text{corners of squares in } F_H\} \\ &= \chi(F_H) + \frac{1}{2}\#\{\text{edges of squares in } F_H \text{ contained in } \partial F_H\} - V + \sum_{p \text{ corner in } F_H} \frac{n_p}{4}, \end{aligned} \tag{11}$$

where n_p is the number of squares that share the corner p. In the above expression, it should be noted that a square in F_H is meant to be a square $S^k_{i,j}$, with $1 \leq i, j \leq n$, and $1 \leq k \leq H_{i,j}$. Rewriting this equation, we find by Lemma 1.10 that

$$\begin{aligned}
\chi(F_H) &= \left(\frac{1}{2}\#\{\text{edges of squares in } F_H \text{ contained in } \partial F_H\} - V + \sum_{p \text{ corner in } F_H} \frac{n_p}{4}\right) \\
&= \sum_{p \text{ corner of a square in } F_H} (1 - \frac{n_p}{4}) - \frac{1}{2}\#\{\text{edges of squares in } F_H \text{ contained in } \partial F_H\} \\
&= \sum_{p \text{ corner of a square in } \partial F_H} (1 - \frac{n_p}{4}) - \frac{1}{2}\#\{\text{edges of squares in } F_H \text{ contained in } \partial F_H\} \\
&= \sum_{p \text{ corner of a square in } \partial F_H} (1 - \frac{n_p}{4}) - \frac{1}{2}\#\{\text{corner of a square in } \partial F_H\} \\
&= \sum_{p \text{ corner of a square in } \partial F_H} (\frac{1}{2} - \frac{n_p}{4}),
\end{aligned} \tag{12}$$

where the penultimate equality follows from the fact that the boundary of F_H is a knot, and that implies that its Euler characteristic is zero, and thus that ∂F_H has as many vertices as edges. By the close inspection of the corner types in Lemma 1.10, we find that for all the types for which there is no O or X at the centre p of the four involved corners, we have that $n_p = 2$, and denoting the corners of O and X by $p_1, \ldots, p_{2n}$ (possibly counting some points multiple times), this gives us that

$$\chi(F_H) = \sum_{p \text{ corner of a square in } \partial F_H} (\frac{1}{2} - \frac{n_p}{4}) = \sum_{i=1}^{2n} (\frac{1}{2} - \frac{n_{p_i}}{4}) = n - \sum_{i=1}^{2n} \frac{n_{p_i}}{4}. \tag{13}$$

For the types for which there is no O or X at the centre p of the four involved corners, we find by close inspection as in Lemma 1.10 that p is a corner of all the squares at the four involved corners, which yields

$$\chi(F_H) = n - \sum_{i=1}^{2n} \frac{n_{p_i}}{4} = n - \theta(H), \tag{14}$$

which gives us in its turn that

$$g(F_H) = 1 - \frac{1}{2}\chi(\Sigma) - \frac{1}{2}b(\Sigma) = \frac{1}{2} - \frac{1}{2}\chi(\Sigma) = \frac{1}{2} - \frac{1}{2}(n - \theta(H)) = \frac{1}{2}\theta(H) + \frac{n-1}{2}. \qquad \square$$

Returning to Example 1.5, it is easy to calculate now that $g(F_{W(\mathbb{G})}) = \frac{1}{2}\frac{24}{4} + \frac{5-1}{2} = 3 + 2 = 5$. The question now remains if this answer depends on our choice of minimal matrix. The answer is no, and we will prove that in the corollary below.

Corollary 1.12 *Let H and H' be two minimal matrices related to $W(\mathbb{G})$. Then $g(F_H) = g(F_{H'})$.*

Proof. There are $s_k, t_k \in \mathbb{Z}$ such that

$$H' - H = \sum_{k=1}^{n} (s_k R_k + t_k C_k). \tag{15}$$

Then note that

$$c(H') - c(H) = c(H' - H) = c(\sum_{k=1}^{n}(s_k R_k + t_k C_k)) = \sum_{k=1}^{n}(s_k n + t_k n) = n\sum_{k=1}^{n}(s_k + t_k). \quad (16)$$

As H and H' are both minimal, we must have by definition that there complexities are equal and thus that

$$0 = c(H') - c(H) = n\sum_{k=1}^{n}(s_k + t_k), \quad (17)$$

that is, $\sum_{k=1}^{n}(s_k + t_k) = 0$. As $\theta(C_k) = 2 = \theta(R_k)$ for all $1 \leq k \leq n$, we now find that

$$\begin{aligned}\theta(H') - \theta(H) = \theta(H' - H) = \theta(\sum_{k=1}^{n}(s_k R_k + t_k C_k)) &= \sum_{k=1}^{n}(s_k \theta(R_k) + t_k \theta(C_k)) \\ = \sum_{k=1}^{n}(s_k \cdot 2 + t_k \cdot 2) = 2\sum_{k=1}^{n}(s_k + t_k),\end{aligned} \quad (18)$$

which proves by Theorem 1.8 that

$$g(F_H) = \frac{1}{2}\theta(H) + \frac{n-1}{2} = \frac{1}{2}\theta(H') + \frac{n-1}{2} = g(F_{H'}).$$ □

Now we have defined a quantity $g(F_H)$ which only depends on the choice of the underlying grid diagram $\mathbb{G}$. We denote this quantity, the associative genus, by $g(\mathbb{G})$. However, this associative genus is not a knot invariant. It is an exercise to find an example of a knot and two different grid diagrams for which the associative genera do not coincide. There is, however, one positive note: we can find the maximal value of $g(\mathbb{G})$ among all grid diagram $\mathbb{G}$ of a knot K. The following theorem says that the associative genus of a knot is smaller than or equal to the Seifert genus of a knot, but also that for every knot, there is always a grid diagram such that associative genus equals the Seifert genus of the knot.

Proposition 1.13 *Let K be a knot and g its Seifert genus. We then have that $g(\mathbb{G}) \leq g$ for any grid diagram $\mathbb{G}$ of K. Moreover, there exists a grid diagram $\mathbb{G}'$ of K such that $g(\mathbb{G}') = g$.*

Proof. Let F be a Seifert surface of K. By applying suitable isotopies, we can view F as a disk with some handles. As F is orientable, we can then further isotopies to make sure that there is no twist in F. We can further apply isotopies to make the boundaries of the handle vertical. In this fashion, we obtain a surface that consists of squares with some edges identified. This surface gives rise to a grid diagram $\mathbb{G}$ of K and to a matrix H' such that the entries display the amount of squares which are projected onto each other. It is an exercise to show that H' is related to $W(\mathbb{G})$ and that $g(F) = \frac{1}{2}\theta(H') - \frac{n-1}{2}$. Now let H be a minimal matrix related to $W(\mathbb{G})$. Then there are $s_k, t_k \in \mathbb{Z}$ such that

$$H' - H = \sum_{k=1}^{n}(s_k R_k + t_k C_k). \quad (19)$$

By the minimality of H, we furthermore know that $\sum_{k=1}^{n}(s_k + t_k) \geq 0$. As $\theta(C_k) = 2 = \theta(R_k)$

for all $1 \leq k \leq n$, we now find that

$$\begin{aligned}\theta(H') - \theta(H) = \theta(H' - H) = \theta(\sum_{k=1}^{n}(s_k R_k + t_k C_k)) &= \sum_{k=1}^{n}(s_k \theta(R_k) + t_k \theta(C_k)) \\ = \sum_{k=1}^{n}(s_k \cdot 2 + t_k \cdot 2) = 2\sum_{k=1}^{n}(s_k + t_k) \geq 0,\end{aligned} \tag{20}$$

which proves by Theorem 1.8 that

$$g(F_H) = \frac{1}{2}\theta(H) + \frac{n-1}{2} \leq \frac{1}{2}\theta(H') + \frac{n-1}{2} = g(F). \tag{21}$$

This proves the first part of the theorem. For the second part, replace F by a Seifert surface for which we have that $g(F)$ is the Seifert genus of K. We then find that $g(F) = g \leq g(F_H)$. Combining both inequalities gives us that $g(F_H) = g$. □

1.3 The genus bound

We will now use this knowledge of the previous subsection to find a bound of the simply blocked grid homology with respect to its Alexander degree.

Lemma 1.14 *Let $\mathbb{G}$ be a grid diagram of a knot K. Then $g(\mathbb{G})$ is the minimum value of the Alexander function over all grid states for $\mathbb{G}$.*

Proof. Let $p_1 = (i_1, j_1), \ldots, p_{8n} = (i_{8n}, j_{8n})$ be the corners of the grid diagram of the squares containing an O or X (possibly counting point multiple times). For an integral matrix B, we then define the function

$$\begin{aligned}A_B : \{\text{grid states of } \mathbb{G}\} &\to \mathbb{Z} \\ \mathbf{x} &\mapsto -\sum_{(i,j)\in\mathbf{x}} B_{i,j} + \frac{1}{8}\sum_{k=1}^{8n} B_{i_k,j_k} - (\frac{n-1}{2}).\end{aligned} \tag{22}$$

Recall that this is the Alexander function if we choose B to be the winding matrix $W := W(\mathbb{G})$ of the grid diagram $\mathbb{G}$. Now note that $A_B = A_{B+R_m} = A_{B+C_m}$ for $1 \leq m \leq n$, since we have that

$$A_{B+R_m}(\mathbf{x}) - A_B(\mathbf{x}) = -\#(\mathbf{x} \cap R_m) + \frac{1}{8}\sum_{k=1}^{8n}\#(p_k \cap R_m) = -1 + \frac{1}{8}\cdot 8 = -1 + 1 = 0,$$

as there are eight p_k in every row (2 for every bordering X, of which there are 2, and 2 for every bordering O, of which there are 2). For the columns, the argument works analogously. Note that imn the context of the intersections we abuse the notation R_k and C_k for $R_k = \{(z,k) \mid z \in \mathbb{Z}\}$ and $C_k = \{(k,z) \mid z \in \mathbb{Z}\}$.
Now let H be a minimal matrix of $W := W(\mathbb{G})$. We then find that $A_H = A_W$, which tells us that A_H is the Alexander function. Note for a grid state $\mathbf{x}$ of $\mathbb{G}$, we have that

$$A_H(\mathbf{x}) = -\sum_{(i,j)\in\mathbf{x}} H_{i,j} + \frac{1}{8}\sum_{k=1}^{8n} H_{i_k,j_k} - (\frac{n-1}{2}). \tag{23}$$

As the last two terms are fixed, the only term we can change is the first term, which we want to be a large as possible in order to find a maximum. As $H_{i,j} \geq 0$ for all $1 \leq i,j \leq n$, we have that $-\sum_{(i,j)\in\mathbf{x}} H_{i,j} \leq 0$. Moreover, by Proposition 1.3, we can find a grid state $\tilde{\mathbf{x}}$ of $\mathbb{G}$ such that $H_{i,j} = 0$ for all $(i,j) \in \tilde{\mathbf{x}}$, which gives that $-\sum_{(i,j)\in\tilde{\mathbf{x}}} H_{i,j} = 0$ and thus that the maximum of the Alexander function $A_W = A_H$ is given by

$$\begin{aligned} A_H(\tilde{\mathbf{x}}) &= -\sum_{(i,j)\in\tilde{\mathbf{x}}} H_{i,j} + \frac{1}{8}\sum_{k=1}^{8n} H_{i_k,j_k} - (\frac{n-1}{2}) \\ &= \frac{1}{2}\sum_{k=1}^{8n} \frac{1}{4} H_{i_k,j_k} - (\frac{n-1}{2}) = \frac{1}{2}\theta(H) - (\frac{n-1}{2}) = g(H). \end{aligned}$$

□

We conclude with our final theorem, which states the genus bound we have been working towards.

Theorem 1.15 *Let K be a knot. We then have that* $\max\{s \mid \widehat{GH}(K,s) \neq 0\} \leq g(K)$.

Proof. By Sven's talk, we know that $\widehat{GH}(K,s)$ is (isomorphic to) a tensor product of $\widetilde{GH}(K,s)$ and some bigraded vector space with no elements with a positive Alexander degree. This gives us that

$$\max\{s \mid \widehat{GH}(K,s) \neq 0\} = \max\{s \mid \widetilde{GH}(\mathbb{G},s) \neq 0\}. \tag{24}$$

By the properties of homology, we also have that

$$\max\{s \mid \widetilde{GH}(K,s) \neq 0\} \leq \max\{s \mid \widetilde{GC}(\mathbb{G},s) \neq 0\}. \tag{25}$$

As $\widetilde{GC}(\mathbb{G},s)$ is a quotient of a vector space generated by grid states of $\mathbb{G}$ with Alexander degree s, we find the maximal value of s for which $\widetilde{GC}(\mathbb{G},s) \neq 0$ is equal to the maximal value of the Alexander function, which is $g(K)$. □

This theorem shows that there is a relatively simple calculation one can do to determine the maximal Alexander degree of the simply blocked grid homology. In a short time, this can give a first idea of the grid homology of a knot, which can come in handy from time to time.

2 Exercises

1. Prove that the relation $\sim$ from Definition 1.1 is an equivalence relation on $M_n(\mathbb{Z})$.

2. Find a minimal matrix related to

$$A = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -2 & -2 & -1 & -1 & -1 & 0 & 1 \\ 0 & -1 & -2 & -2 & -1 & -1 & 0 & 1 & 1 \\ 0 & -1 & -2 & -2 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

or program an algorithm to do it for you.

3. Prove Proposition 1.3.
4. Prove the following statement: if $A, A' \in M_N(\mathbb{N})$ are minimal and if $A \sim A'$, then $\operatorname{tr}(A) = \operatorname{tr}(A')$.
5. Prove Proposition 1.7. Hint: you might want ot use Proposition 1.6 and the following facts:
   - for fixed j, there is are two i for which $|H_{i,j} - H_{i,j+1} - H_{i+1,j} + H_{i+1,j+1}| = 1$ holds;
   - for fixed j, there are $1 \leq k, l \leq n$ such that $H_{k,j} = 0 = H_{l,j+1}$;
   - all entries of H are non-negative.
6. Prove Theorem 1.8.
7. Find two grid diagrams $\mathbb{G}$ and $\mathbb{G}'$ of the same knot K such that $g(\mathbb{G}) \neq g(\mathbb{G}')$.
8. Proof that for the Seifert surface F and its corresponding (projection) matrix H' of Proposition 1.13 satisfy $g(F) = \frac{1}{2}\theta(H') - \frac{n-1}{2}$ and show further that H' is related to the winding matrix $W(\mathbb{G})$.
9. Let K be the figure-eight knot. Compute the associative genus of the following grid diagram of K. Also compute the Seifert genus of K, and compare both results to the simply blocked grid homology $\widehat{GH}(K, s)$.

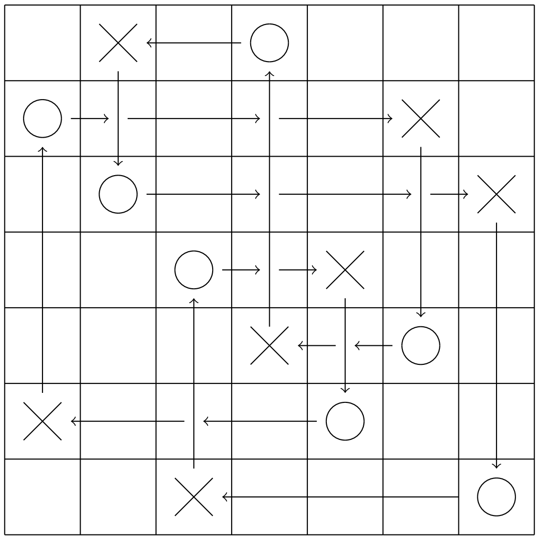

Figure 5: A grid diagram for the figure-eight knot.