Topics in Topology Lecture 7

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15 March 2021

A genus bound

Theorem

Let K be a knot. We then have that $\max\{s \mid \widehat{GH}(K, s) \neq 0\} \leq g(K)$.

Definitions of some types of matrices

- **Integral matrices:**
  $M_n(\mathbb{Z}) := \{A = (A_{i,j}) \mid A_{i,j} \in \mathbb{Z} \text{ for all } 1 \leq i, j \leq n\}$.
- $M_n(\mathbb{N}) := \{A = (A_{i,j}) \mid A_{i,j} \in \mathbb{N} \text{ for all } 1 \leq i, j \leq n\}$.
- C_k is the matrix with 1's in column k and zeroes elsewhere
- R_k is the matrix with 1's in row k and zeroes elsewhere

Definition

Let $A, B, \in M_n(\mathbb{Z})$.
$A \sim B$ if and only if $A - B = \sum_{k=1}^{n}(s_k R_k + t_k C_k)$ for $s_k, t_k \in \mathbb{Z}$.

A minimal matrix

Definition

The **complexity** of $A \in M_n(\mathbb{Z})$ is $c(A) := \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j}$. A matrix $A \in M_n(\mathbb{N})$ is **minimal** if and only if for all $B \in M_n(\mathbb{N})$ such that $A \sim B$, we have that $c(B) \geq c(A)$.

A criterion for minimality (part 1)

Proposition

Let $A \in M_n(\mathbb{N})$. Then A is minimal if and only if there exists a permutation $\sigma \in S_n$ such that $A_{k,\sigma(k)} = 0$ for all $1 \leq k \leq n$.

Proof.

A criterion for minimality (part 2)

Proof. (Continued)

The running example (part 1)

Example

Let $\mathbb{G}$ be given by Figure 1.

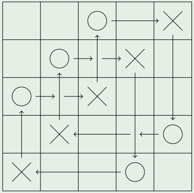

Figure: A grid diagram for the left handed trefoil knot.

Then the winding matrix $W(\mathbb{G})$ is given by

$$W(\mathbb{G}) := \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which is already minimal.

Properties of a minimal matrix (part 1)

Let H be a minimal matrix related to $W(\mathbb{G})$.

Proposition

For $1 \leq i, j \leq n$, we have that

$$|H_{i,j} - H_{i,j+1} - H_{i+1,j} + H_{i+1,j+1}| \leq 1, \tag{1}$$

where addition is taken modulo n.

Proof.

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Properties of a minimal matrix (part 2)

Proof. (Continued)

Properties of a minimal matrix (part 3)

Proposition

For $1 \leq i, j \leq n$, we have that

$$|H_{i,j} - H_{i,j+1}| \leq 1 \tag{2}$$

and that

$$|H_{i,j} - H_{i+1,j}| \leq 1, \tag{3}$$

where addition is taken modulo n.

The construction of F_H.

Construct the surface F_H as follows. First, for every $1 \leq i, j \leq n$, create $H_{i,j}$ squares, and call such a square $s_{i,j}^k$. (If $H_{i,j} = 0$, do not create any squares.) Then glue by the following rules:

- Glue the right edge of $s_{i,j}^k$ to the left edge of $s_{i,j+1}^k$ for $1 \leq k \leq \min(H_{i,j}, H_{i,j+1})$.
- Glue the bottom edge $s_{i,j}^{H_{i,j}-k}$ to the top edge of $s_{i+1,j}^{H_{i+1,j}-k}$ for $0 \leq k \leq \min(H_{i,j}, H_{i,j+1}) - 1$.

The running example (part 2)

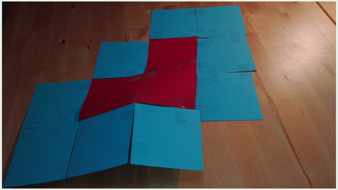
Example

The winding matrix $W(\mathbb{G})$ of the discussed grid diagram of the trefoil knot is given by

$$W(\mathbb{G}) := \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which is already minimal.

Then the corresponding Seifert surface $F_{W(\mathbb{G})}$ is given as follows.



H induces a Seifert surface

Theorem

Let K be knot, $\mathbb{G}$ a grid diagram of K and $W(\mathbb{G})$ its winding matrix. If H is a minimal matrix related to $W(\mathbb{G})$, then F_H is a Seifert surface.

Genus

Definition

Let Σ be a 2-dimensional CW-complex (or a 2-dimensional surface). Then the **genus** of Σ is given by

$$g(\Sigma) := 1 - \frac{1}{2}\chi(\Sigma) - \frac{1}{2}b(\Sigma), \tag{4}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ, and $b(\Sigma)$ its number of boundary components.

The **Seifert genus** of a knot K is the minimal value of the genus of any Seifert surface of K.

A property of F_H (part 1)

Lemma

Let H be a minimal matrix related to the winding matrix $W(\mathbb{G})$ of a grid diagram $\mathbb{G}$. If $x \in \mathrm{int}(F_H)$ is a corner point of a square $s^k_{i,j}$, then x is a corner of exactly four squares $s^k_{i,j}$.

Proof.

A property of F_H (part 2)

Proof. (Continued)

The genus of F_H (part 1)

Theorem

The genus of F_H is given by

$$g(F_H) = \frac{1}{2}\theta(H) + \frac{n-1}{2}, \tag{5}$$

where $\theta(H)$ is the sum of the averages of the corners around each O and X.

Proof.

The genus of F_H (part 2)

Proof. (Continued)

The running example (part 3)

Example

Let $\mathbb{G}$ be given by Figure 2.

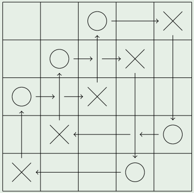


Figure: A grid diagram for the left handed trefoil knot.

What is $g(F_{W(\mathbb{G})})$?

Then the winding matrix $W(\mathbb{G})$ is given by

$$W(\mathbb{G}) := \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which is already minimal.

Grid diagram invariancy

Corollary

Let H and H' be two minimal matrices related to $W(\mathbb{G})$. Then $g(F_H) = g(F_{H'})$.

Proof.

Associated genus and Seifert genus (part 1)

Definition

Let $g(\mathbb{G})$ be a grid diagram. Then the **associated genus** of $\mathbb{G}$ is defined by $g(\mathbb{G}) := g(F_H)$, where H is a minimal matrix of $W(\mathbb{G})$.

Associated genus and Seifert genus (part 2)

Proposition

Let K be a knot and g its Seifert genus. We then have that $g(\mathbb{G}) \leq g$ for any grid diagram $\mathbb{G}$ of K. Moreover, there exists a grid diagram $\mathbb{G}'$ of K such that $g(\mathbb{G}') = g$.

Proof. (Sketch)

- F Seifert surface of $K \overset{\text{isotopies}}{\rightsquigarrow}$ union of squares
- $\rightsquigarrow$ grid diagram $\mathbb{G}$ and "projection matrix" with nice properties
- $\rightsquigarrow$ Genus of $F \leq$ associated genus of $\mathbb{G}$
- Repeat for Seifert surface with minimal genus

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The maximum for the Alexander function (part 1

Lemma

Let $\mathbb{G}$ be a grid diagram of a knot K. Then $g(\mathbb{G})$ is the minimum value of the Alexander function over all grid states for $\mathbb{G}$.

The maximum for the Alexander function (part 2)

Proof. (Continued)

The final theorem

Theorem

Let K be a knot. We then have that $\max\{s \mid \widehat{GH}(K,s) \neq 0\} \leq g(K)$.

More frames!

More frames!

More frames!

More frames!

The figure-eight knot