Stabilization Invariance of Grid Homology—Topics in

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Abstract

These lecture notes are meant as a hand-out for my talk on the stabilization invariance of grid homology given on the 1st of March 2021 during the Topology seminar hosted by Roland van der Veen and Jorge Becerra. In these notes we will prove that the grid homology $\widehat{\mathrm{GH}}(\mathbb{G})$ is invariant under stabilization moves for the grid diagram $\mathbb{G}$. I am a rather algebraically minded person and I have therefore decided to take a more algebraic approach. Due to this decision the prerequisites might be a bit long, but this hard work will pay off later.

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1 Introduction

Let $\mathbb{G}$ be a grid diagram representing a knot and let $\mathbb{G}'$ be the grid diagram obtained from $\mathbb{G}$ by a stabilization move. In these lecture notes we will show that the grid homology $\widehat{\mathrm{GH}}$ is invariant under stabilization moves, i.e. $\widehat{\mathrm{GH}}(\mathbb{G}') \cong \widehat{\mathrm{GH}}(\mathbb{G})$. In order to do so we will need quite some tools from commutative algebra and homological algebra. The next chapter will be dedicated to these needed tools. However, I will assume that the reader is at least somewhat familiar with most of the material from chapters 1,2,3 and 4 of [2].

2 Prerequisites

In order to proof that $\widehat{\mathrm{GH}}$ is invariant under stabilization we need a convenient language. This section provides the required machinery. We start be stating some results regarding tensor products and exact sequences.

Definition 2.1. *Let* $X = \bigoplus_{d,s\in\mathbb{Z}} X_{d,s}$ *and* $Y = \bigoplus_{d,s\in\mathbb{Z}} Y_{d,s}$ *be two bigraded vector spaces. Their tensor product* $X \otimes Y = \bigoplus_{d,s\in\mathbb{Z}} (X \otimes Y)_{d,s}$ *is the bigraded vector space with* $(X \otimes Y)_{d,s} = \bigoplus_{\substack{d_1+d_2=d \\ s_1+s_2=s}} (X_{d_1,s_1} \otimes Y_{d_2,s_2})$.

For readers uncomfortable with tensor products I highly recommend the notes from Keith Conrad [1].

Definition 2.2. *Let X be a bigraded vector space, and fix integers a and b. The corresponding shift of X, denoted $X[\![a,b]\!]$, is the bigraded vector space that is isomorphic to X as a vector space and given the bigrading $X[\![a,b]\!]_{d,s} = X_{d+a,s+b}$.*

An important property regarding this shift is that it can be related to the tensor product.

Lemma 2.3. *Let W be the bigraded vector space of dimension 2 with 1 generator in bigrading $(0,0)$ and 1 generator in bigrading $(-1,-1)$. Let X be any other bigraded vector space, then $X \otimes W \cong X \oplus X[\![1,1]\!]$.*

Proof. Write $W = W_{0,0} \oplus W_{-1,-1}$, then $X \otimes W = (X \otimes W_{0,0}) \oplus (X \otimes W_{-1,-1})$. The result now follows from Definitions 2.1 and 2.2 (see Exercise 4.2). $\square$

Definition 2.4. *Let $\{V_i\}_{i=0}^n$ be vector spaces over some field and consider the sequence*

$$V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} V_2 \xrightarrow{f_3} V_3 \xrightarrow{f_4} \cdots \xrightarrow{f_n} V_n$$

of linear maps between them. We say that this sequence is exact if $\mathrm{im}(f_k) = \ker(f_{k+1})$ for all $k = 1, \ldots, n-1$.

Definition 2.5. *A short exact sequence of vector spaces is a sequence of the form*

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0,$$

that is exact.

An important property of short exact sequences of vector spaces is that it gives us a decomposition. In fact, consider the short exact sequence as given in Definition 2.5. Then we have $V \cong U \oplus W$ as vector spaces (see Exercise 4.1). Note that this is also true for infinite dimensional vector spaces, as vector spaces are always free (have a basis) when the Axiom of Choice is assumed. We will need this decomposition for the proof of Proposition 3.1.

The rest of the prerequisites is dedicated to some more homological algebra. The reader uncomfortable with modules over a ring can just think of them as vector spaces over a field, as we will only need this machinery for vector spaces in chapter 3. We follow closely [2, Appendix A]. Let $\mathbb{K}$ denote either the finite field $\mathbb{Z}/p\mathbb{Z}$ for some prime $p \in \mathbb{N}$, or $\mathbb{Q}$ or the ring $\mathbb{Z}$. In the following, $\mathcal{R}$ will denote the polynomial ring $\mathbb{K}[V_1, \dots, V_n]$ of n variables. We include the $n = 0$ case with the understanding that in this case $\mathcal{R} = \mathbb{K}$. In particular, $\mathcal{R}$ is a field if $n = 0$ and $\mathbb{K}$ is not $\mathbb{Z}$; and $\mathcal{R}$ is a principal ideal domain (PID) if either $n = 0$, or $n = 1$ and $\mathbb{K}$ is not $\mathbb{Z}$.

Definition 2.6. *A chain complex is an $\mathcal{R}$-module C, equipped with an $\mathcal{R}$-module homomorphism $\partial\colon C \to C$ with the property that $\partial^2 = 0$. The map ∂ is called the boundary map or the differential for C. A cycle is an element $z \in C$ with $\partial z = 0$ and a boundary is an element b of the form $b = \partial a$ for some $a \in C$; i.e. the cycles are the elements in the kernel* $\operatorname{Ker}\partial$, *and the boundaries are the elements in the image* $\operatorname{Im}\partial$.

Definition 2.7. *An $\mathcal{R}$-submodule $C' \subset C$ of a chain complex (C, ∂) is a subcomplex if $\partial(C') \subset C'$. In this case, the pair $(C', \partial_{|C'})$ is a chain complex. Similarly, if $(C', \partial_{|C'})$ is a subcomplex, the quotient module C/C' inherits a boundary operator $\partial_{C/C'}$, induced from ∂. The pair $(C/C', \partial_{C/C'})$ is the quotient complex of C by C'.*

Note that the condition $\partial^2 = 0$ implies that $\operatorname{Im}\partial \subset \operatorname{Ker}\partial$, which gives us the next definition.

Definition 2.8. *The homology $H(C, \partial)$ of the chain complex (C, ∂) is the quotient $\mathcal{R}$-module* $\operatorname{Ker}\partial / \operatorname{Im}\partial$.

Definition 2.9. *A bigraded chain complex over $\mathcal{R}$ is a chain complex (C, ∂) over $\mathbb{K}$, equipped with endomorphisms $V_i\colon C \to C$ for $i = 1, \dots, n$, called the algebra actions, and a splitting $C = \bigoplus_{d,s \in \mathbb{Z}} C_{d,s}$, satisfying the following compatibility conditions:*

- *for $i = 1, \dots, n$, $\partial \circ V_i = V_i \circ \partial$;*
- *for all $i, j \in \{1, \dots, n\}$, $V_i \circ V_j = V_j \circ V_i$;*
- *∂ maps $C_{d,s}$ to $C_{d-1,s}$;*
- *V_i maps $C_{d,s}$ to $C_{d-2,s-1}$.*

The first grading is called the Maslov grading, and the second the Alexander grading.

Definition 2.10. *Let (C, ∂) and (C', ∂') be two chain complexes over $\mathcal{R}$. An $\mathcal{R}$-module map $f\colon C \to C'$ is a chain map if it commutes with the boundary operators, that is, for every $c \in C$, $f(\partial c) = \partial' f(c)$.*

Definition 2.11. *If (C, ∂) and (C', ∂') are two bigraded chain complexes over $\mathcal{R}$, then a bigraded chain map f between them is a chain map that maps $C_{d,s}$ to $C'_{d,s}$. More generally, if C and C' are as above, $f\colon C \to C'$ is a chain map, and there is a pair of integers (m, t) so that f maps $C_{d,s}$ into $C_{d+m,s+t}$, then we say that f is a homogeneous map with bidegree (m, t).*

Lemma 2.12. *Let (C, ∂) and (C', ∂') be two chain complexes over $\mathcal{R}$ and let f be a chain map between them. Then f induces an $\mathcal{R}$-module map $H(f)\colon H(C, \partial) \to H(C', \partial')$ on the homology modules. When C and C' are bigraded complexes over $\mathcal{R}$ and $f\colon C \to C'$ is a bigraded chain map, the induced map $H(f)$ respects the induced bigrading on the homology modules.*

Proof. See exercise 4.3. □

Definition 2.13. *An exact sequence of chain complexes over $\mathcal{R}$ is an exact sequence of $\mathcal{R}$-modules such that the maps are also chain maps. An exact sequence of bigraded chain complexes over $\mathcal{R}$ is an exact sequence of bigraded $\mathcal{R}$-modules such that the maps are also bigraded chain maps.*

A very important result in homological algebra that we will need, but not prove in full detail, is the following.

Lemma 2.14. *To each short exact sequence of chain complexes of $\mathcal{R}$-modules*

$$0 \longrightarrow (C, \partial) \xrightarrow{\;f\;} (C', \partial') \xrightarrow{\;g\;} (C'', \partial'') \longrightarrow 0,$$

there is an associated $\mathcal{R}$-module homomorphism $\delta : H(C'', \partial'') \to H(C, \partial)$, called the connecting homomorphism, such that

$$\begin{tikzcd} & H(C,\partial) \arrow[dr, "H(f)"] & \\ H(C'',\partial'') \arrow[ur, "\delta"] & & H(C',\partial') \arrow[ll, "H(g)"] \end{tikzcd}$$

is an exact triangle. Moreover, if the three complexes are bigraded complexes of $\mathcal{R}$-modules, and f and g are homogeneous $\mathcal{R}$-module homomorphisms of degree (m_1, t_1) and (m_2, t_2) respectively, then δ is a homogeneous map with degree $(-m_1 - m_2 - 1, -t_1 - t_2)$.

Proof. For the sake of this hand-out we only care about how δ is constructed. Details on well-definedness and existence of elements are omitted. Given a homology class $x \in H(C'', \partial'')$, define $\delta(x)$ as follows. Pick a cycle $c'' \in C''$ representing x, and find $c \in C$ and $c' \in C'$ so that $g(c') = c''$ and $f(c) = \partial' c'$. We then define $\delta(x)$ to be the homology class represented by c. In the bigraded case, if c'' is supported in bigrading (d, s), then we can find c' in bigrading $(d - m_2, s - t_2)$, and hence c in bigrading $(d - m_1 - m_2 - 1, s - t_1 - t_2)$. For further details see [2, Lemma A.2.1] and exercise 4.4. □

It is important to note that, in general, we do not get a short exact sequence. In fact, there is no reason for $H(f)$ to be injective.

In the proof of Corollary 3.3 we have to show that certain bigraded vector spaces are isomorphic. Just as a finite dimensional vector space is determined up to isomorphism by its dimension, a finite dimensional bigraded vector space Y is determined up to isomorphism by its Poincaré polynomial P_Y.

Definition 2.15. *The Poincaré polynomial P_Y of a bigraded vector space Y is the Laurent polynomial in q and t:*

$$P_Y(q, t) \coloneqq \sum_{d,s \in \mathbb{Z}} \dim Y_{d,s} \cdot q^d t^s.$$

An important fact that we will use is that the Poincaré polynomial is multiplicative with respect to the tensor product of two bigraded vector spaces.

3 Main Results

In this section we will prove that $\widehat{\mathrm{GH}}$ is invariant under stabilization. Let $\mathbb{G}$ be a grid diagram representation a knot and let $\mathbb{G}'$ be the grid diagram obtained after a stabilization move.

Proposition 3.1. *There is an isomorphism of bigraded vector spaces*

$$\widetilde{\mathrm{GH}}(\mathbb{G}') \cong \widetilde{\mathrm{GH}}(\mathbb{G}) \oplus \widetilde{\mathrm{GH}}(\mathbb{G})[\![1,1]\!].$$

Proposition 3.2. *Let W be the 2-dimensional vector with one generator in bigrading $(0,0)$ and one generator in bigrading $(-1,-1)$. then there is an isomorphism of bigraded vector spaces*

$$\widetilde{\mathrm{GH}}(\mathbb{G}) \cong \widehat{\mathrm{GH}}(\mathbb{G}) \otimes W^{\otimes(n-1)},$$

where n is the grid number of $\mathbb{G}$.

Proof. See [2, Proposition 4.6.15]. □

As a corollary of these propositions we obtain that $\widehat{GH}$ is invariant under stabilization.

Corollary 3.3. *The simply blocked grid homology, that is $\widehat{\mathrm{GH}}$, is invariant under stabilization. More formally, we have the isomorphism of bigraded vector spaces $\widehat{\mathrm{GH}}(\mathbb{G}') \cong \widehat{\mathrm{GH}}(\mathbb{G})$.*

Proof. By Proposition 3.1 we have $\widetilde{\mathrm{GH}}(\mathbb{G}') \cong \widetilde{\mathrm{GH}}(\mathbb{G}) \oplus \widetilde{\mathrm{GH}}(\mathbb{G})[\![1,1]\!]$ and using Lemma 2.3 we find that

$$\widetilde{\mathrm{GH}}(\mathbb{G}') \cong \widetilde{\mathrm{GH}}(\mathbb{G}) \otimes W. \tag{3.1}$$

On the other hand we obtain from Proposition 3.2 that

$$\widetilde{\mathrm{GH}}(\mathbb{G}') \cong \widehat{\mathrm{GH}}(\mathbb{G}') \otimes W^{\otimes n} \tag{3.2}$$

and that

$$\widetilde{\mathrm{GH}}(\mathbb{G}) \cong \widehat{\mathrm{GH}}(\mathbb{G}) \otimes W^{\otimes(n-1)}. \tag{3.3}$$

Plugging equation (3.3) into (3.1) yields

$$\widetilde{\mathrm{GH}}(\mathbb{G}') \cong \widehat{\mathrm{GH}}(\mathbb{G}) \otimes W^{\otimes n}.$$

From (3.2) we then obtain that

$$\widehat{\mathrm{GH}}(\mathbb{G}) \otimes W^{\otimes n} \cong \widehat{\mathrm{GH}}(\mathbb{G}') \otimes W^{\otimes n}.$$

We would like to be able to 'cancel' the $W^{\otimes n}$ on both side to get the desired isomorphism. Luckily $\widehat{\mathrm{GH}}(\mathbb{G})$, $\widehat{\mathrm{GH}}(\mathbb{G}')$ and W are all finite dimensional bigraded vector spaces (see [2, Corollary 4.6.16]), so we only need to show that the Poincaré polynomials of $\widehat{\mathrm{GH}}(\mathbb{G})$ and $\widehat{\mathrm{GH}}(\mathbb{G}')$ coincide (see Definition 2.15). We have

$$P_{\widehat{\mathrm{GH}}(\mathbb{G})\otimes W^{\otimes n}}(q,t) = P_{\widehat{\mathrm{GH}}(\mathbb{G})}(q,t) \cdot P_W(q,t)^n = P_{\widehat{\mathrm{GH}}(\mathbb{G})}(q,t) \cdot (1+q^{-1}t^{-1})^n$$

and

$$P_{\widehat{\mathrm{GH}}(\mathbb{G}')\otimes W^{\otimes n}}(q,t) = P_{\widehat{\mathrm{GH}}(\mathbb{G}')}(q,t) \cdot P_W(q,t)^n = P_{\widehat{\mathrm{GH}}(\mathbb{G}')}(q,t) \cdot (1+q^{-1}t^{-1})^n.$$

These two expressions are equal, hence $P_{\widehat{\mathrm{GH}}(\mathbb{G})}(q,t) = P_{\widehat{\mathrm{GH}}(\mathbb{G}')}(q,t)$, so that the desired isomorphism follows. □

The rest of this section will be dedicated to proving Proposition 3.1. Suppose $\mathbb{G}'$ is obtained by a stabilization of type $X{:}SW$ (see Figure 3.1). It suffices to consider only this case as any other stabilization is obtained from this one using a combination of commutations and switches for which we already know they leave $\widehat{\mathrm{GH}}$ invariant (see [2, Corollary 3.2.3]) and [2, Section 5.1]).

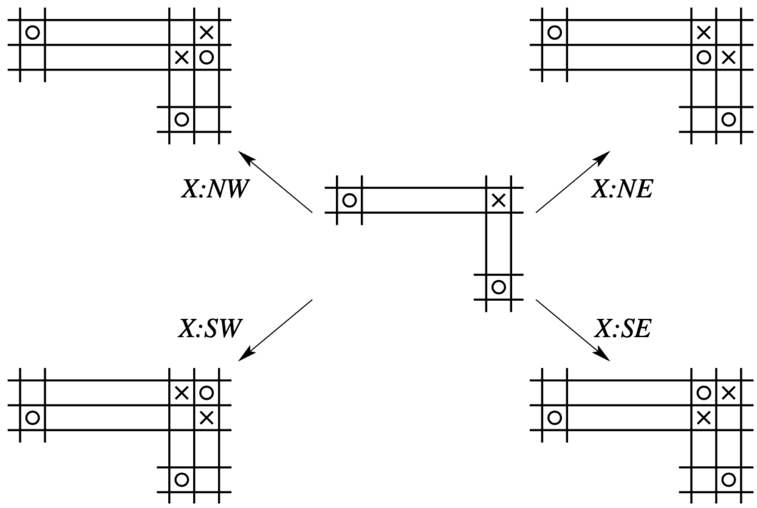

Figure 3.1: The four different type of stabilizations [2, Figure 3.8].

In other words, $\mathbb{G}$ is gotten from $\mathbb{G}'$ by destabilizing at the 2×2 square $\begin{array}{c|c} X & O \\ \hline & X \end{array}$. We number the O-markings so that O_1 is the newly introduced one and O_2 is the O-marking in the row just below O_1. Moreover, let c denote the intersection point of the newly obtained horizontal and vertical circles and lets label the X-markings $\begin{array}{c|c} X_1 & O_1 \\ \hline & X_2 \end{array}$.

We decompose the set of grid states $S(\mathbb{G}')$ of the stabilized diagram $\mathbb{G}'$ as the disjoint union $I(\mathbb{G}') \cup N(\mathbb{G}')$, where $I(\mathbb{G}')$ is the set of grid states $x \in S(\mathbb{G}')$ with $c \in x$. It should be clear that the chain complex $\widetilde{\mathrm{GC}}(\mathbb{G}')$ then decomposes as $\widetilde{\mathrm{GC}}(\mathbb{G}') \cong \widetilde{I} \oplus \widetilde{N}$, where $\widetilde{I} = \operatorname{span} I(\mathbb{G}')$ and $\widetilde{N} = \operatorname{span} N(\mathbb{G}')$ (as vector spaces over $\mathbb{F}_2$).

Lemma 3.4. *The subspace $\widetilde{N}$ is a subcomplex of $\widetilde{\mathrm{GC}}(\mathbb{G}')$.*

Proof. We need to show that $\widetilde{\partial}_{\mathbb{O},\mathbb{X}}(\widetilde{N}) \subset \widetilde{N}$, where the differential is specified by

$$\widetilde{\partial}_{\mathbb{O},\mathbb{X}}(x) = \sum_{y \in S(\mathbb{G}')} \#\{r \in \mathrm{Rect}^\circ(x,y) | r \cap \mathbb{O} = r \cap \mathbb{X} = \emptyset\} \cdot y$$

for $x \in S(\mathbb{G}')$. Indeed, if $y \in N(\mathbb{G}')$ this is clear. Suppose now that $y \in I(\mathbb{G}')$, then any rectangle $r \in \mathrm{Rect}(x, y)$ contains either X_1 or X_2, which means that those terms vanish in the sum. □

The above also tells us that $\widetilde{I} \cong \widetilde{\mathrm{GC}}(\mathbb{G}')/\widetilde{N}$, which will be useful later. Now with respect to the splitting $\widetilde{\mathrm{GC}}(\mathbb{G}') \cong \widetilde{I} \oplus \widetilde{N}$ the map $\widetilde{\partial}_{\mathbb{O},\mathbb{X}}$ can be written in matrix form as

$$\widetilde{\partial}_{\mathbb{O},\mathbb{X}} = \begin{bmatrix} \widetilde{\partial}_I^I & 0 \\ \widetilde{\partial}_I^N & \widetilde{\partial}_N^N \end{bmatrix},$$

where the zero corresponds to the fact that there are no empty rectangles from $\widetilde{N}$ to $\widetilde{I}$ (see proof of Lemma 3.4).

Lemma 3.5. *There is a bijection between $S(\mathbb{G})$ and $I(\mathbb{G}')$. It is given by*

$$f\colon S(\mathbb{G}) \to I(\mathbb{G}'), x \mapsto x \cup \{c\}.$$

Proof. This is clear. □

Lemma 3.6. *For $x \in S(\mathbb{G})$, let $x' = x \cup \{c\} \in I(\mathbb{G}')$. Then $M(x') = M(x) - 1$ and $A(x') = A(x) - 1$, where M and A are the Maslov and Alexander functions on grid states respectively.*

Proof. See [2, Lemma 5.2.4] □

Lemma 3.7. *The map $\widetilde{e} : \widetilde{I} \to \widetilde{\mathrm{GC}}(\mathbb{G})$ induced by the correspondence $x \cup \{c\} \mapsto x$ gives an isomorphism between the bigraded chain complexes $(\widetilde{I}, \widetilde{\partial}_I^I)$ and $\widetilde{\mathrm{GC}}(\mathbb{G})[\![1,1]\!]$.*

Proof. The map $\widetilde{e}$ is a bijection between grid states, so induces an isomorphism between vector spaces (as both of them are generated by grid states over $\mathbb{F}_2$). Clearly empty rectangles disjoint from $\mathbb{O} \cup \mathbb{X}$ in $\mathbb{G}$ correspond to empty rectangles disjoint from $\mathbb{O}' \cup \mathbb{X}'$ in $\mathbb{G}'$. This implies that $\widetilde{e}$ is in fact an isomorphism between chain complexes. The grading follows from Lemma 3.6. □

In order to actually prove Proposition 3.1 we need to relate the homologies of $\widetilde{I}$ and $\widetilde{N}$. Consider the linear map $\widetilde{H}^I_{X_2} \colon \widetilde{N} \to \widetilde{I}$, which maps

$$\widetilde{N}(\mathbb{G}') \ni x \mapsto \sum_{y \in I(\mathbb{G}')} \#\{r \in \mathrm{Rect}^\circ(x,y) | \,\mathrm{Int}(r) \cap \mathbb{O} = \emptyset, \mathrm{Int}(r) \cap \mathbb{X} = X_2\} \cdot y$$

and the linear map $\widetilde{H}_{O_1} \colon \widetilde{I} \to \widetilde{N}$, which maps

$$\widetilde{I}(\mathbb{G}') \ni x \mapsto \sum_{y \in N(\mathbb{G}')} \#\{r \in \mathrm{Rect}^\circ(x,y) | \,\mathrm{Int}(r) \cap \mathbb{X} = \emptyset, \mathrm{Int}(r) \cap \mathbb{O} = O_1\} \cdot y.$$

Lemma 3.8. *The map $\widetilde{H}^I_{X_2}$ drops Maslov grading and Alexander grading by 1, while the map $\widetilde{H}_{O_1}$ increases both gradings by 1. Moreover, the maps $\widetilde{H}^I_{X_2}$ and $\widetilde{H}_{O_1}$ are chain maps, and induce isomorphisms on homology.*

Proof. See [2, Lemma 5.2.6]. □

Lemma 3.9. *The chain map $\widetilde{\partial}_I^N : (\widetilde{I}, \widetilde{\partial}_I^I) \to (\widetilde{N}, \widetilde{\partial}_N^N)$ induces the trivial map on homology.*

Proof. See [2, Lemma 5.2.8]. □

We are now finally ready to prove Proposition 3.1.

Proof of Proposition 3.1. As stated before we only consider the case of a stabilization of the form $X{:}SW$, with $\widetilde{N}$ a subcomplex of $\widetilde{\mathrm{GC}}(\mathbb{G}')$ with quotient $\widetilde{I}$. We obtain the short exact sequence

$$0 \longrightarrow \widetilde{N} \xrightarrow{\ i\ } \widetilde{\mathrm{GC}}(\mathbb{G}') \xrightarrow{\ \pi\ } \widetilde{I} \longrightarrow 0,$$

of chain complexes, where i is injection and π is the quotient map. By Lemma 2.14 we have a connecting homomorphism $\delta \colon H(\widetilde{I}, \widetilde{\partial}_I^I) \to H(\widetilde{N}, \widetilde{\partial}_N^N)$ so that

$$\begin{tikzcd} & H(\widetilde{N}, \widetilde{\partial}_N^N) \arrow[dr, "H(i)"] & \\ H(\widetilde{I}, \widetilde{\partial}_I^I) \arrow[ur, "\delta"] & & \widetilde{\mathrm{GH}}(\mathbb{G}') \arrow[ll, "H(\pi)"] \end{tikzcd}$$

is an exact triangle. This connecting homomorphism coincides with the induced homology map obtained from the chain map $\widetilde{\partial}_I^N : \widetilde{I} \to \widetilde{N}$ (See exercise 4.6). By Lemma 3.9 we then find that δ is the trivial map. Therefore we have the short exact sequence

$$0 \longrightarrow H(\widetilde{N}, \widetilde{\partial}_N^N) \longrightarrow \widetilde{\mathrm{GH}}(\mathbb{G}') \longrightarrow H(\widetilde{I}, \widetilde{\partial}_I^I) \longrightarrow 0,$$

which reduces to the short exact sequence

$$0 \longrightarrow \widetilde{\mathrm{GH}}(\mathbb{G}) \longrightarrow \widetilde{\mathrm{GH}}(\mathbb{G}') \longrightarrow \widetilde{\mathrm{GH}}(\mathbb{G})[\![1,1]\!] \longrightarrow 0$$

due to Lemma 3.7 and Lemma 3.8. The result now follows from exercise 4.1. $\square$

4 Exercises

Exercise 4.1. *Prove the property of short exact sequences of finite dimensional vector spaces mentioned immediately after Definition 2.5 (Hint: For the finite dimensional case you can use the Rank-Nullity theorem).*

Exercise 4.2. *Finish the proof of Lemma 2.3.*

Exercise 4.3. *Prove Lemma 2.12 (Hint: What happens to boundaries and cycles?).*

Exercise 4.4. *Prove that the map δ in Lemma 2.14 is well-defined.*

Exercise 4.5. *Let R be a commutative ring with 1 and let M, N and W be R-modules. Is it always the case that $M \otimes_R W \cong N \otimes_R W$ implies $M \cong N$? (Hint: consider $R = \mathbb{Z}$ and certain quotients of $\mathbb{Z}$). Compare this with the proof of Corollary 3.3.*

Exercise 4.6. *Show that the induced homology map from the chain map $\widetilde{\partial}_I^N : \widetilde{I} \to \widetilde{N}$ indeed coincides with the connecting homomorphism δ from the proof of Proposition 3.1.*

Solution. We have the short exact sequence

$$0 \longrightarrow \widetilde{N} \xrightarrow{\ j\ } \widetilde{I} \oplus \widetilde{N} \xrightarrow{\ \pi\ } \widetilde{I} \longrightarrow 0,$$

where $j(n) = (0, n)$ and $\pi(i, n) = i$. We need to show that $\delta(\bar{z}) = H(\widetilde{\partial}_I^N)(\bar{z})$ for $\bar{z} \in H(\widetilde{I}, \widetilde{\partial}_I^I)$, so let z be a cycle representing the homology class $\bar{z}$. Consider the element $(z, 0) \in \widetilde{I} \oplus \widetilde{N}$ and the element $\widetilde{\partial}_I^N(z) \in (\widetilde{N}, \widetilde{\partial}_N^N)$. Then we have $\pi(z, 0) = z$ and

$$\widetilde{\partial}_{\mathbb{O},\mathbb{X}}(z, 0) \overset{\heartsuit}{=} (\widetilde{\partial}_I^I(z), \widetilde{\partial}_I^N(z)) \overset{\clubsuit}{=} (0, \widetilde{\partial}_I^N(z)) = j(\widetilde{\partial}_I^N(z)),$$

where in $\heartsuit$ we used the matrix characterization of $\widetilde{\partial}_{\mathbb{O},\mathbb{X}}$ and in $\clubsuit$ we used the fact that z is a cycle. From the construction of the connecting homomorphism we then find that $\delta(\bar{z})$ equals the homology class of $\widetilde{\partial}_I^N(z)$, as desired. $\square$

Exercise 4.7. *Consider the following two grid diagrams*

$$\mathbb{G}_1 = \begin{array}{|c|c|}\hline O & X \\ \hline X & O \\ \hline \end{array} \quad \textit{and} \quad \mathbb{G}_2 = \begin{array}{|c|c|c|}\hline & X & O \\ \hline O & & X \\ \hline X & O & \\ \hline \end{array}$$

of the unknot and note that $\mathbb{G}_2$ *is obtained by a stabilization move* X:SW *of* $\mathbb{G}_1$. *Compute* $\widetilde{\mathrm{GH}}(\mathbb{G}_1)$, $\widetilde{\mathrm{GH}}(\mathbb{G}_2)$, $\widehat{\mathrm{GH}}(\mathbb{G}_1)$ *and* $\widehat{\mathrm{GH}}(\mathbb{G}_2)$. *What do you notice?*

References

[1] Keith Conrad. Tensor products. *Notes of course, available on-line*, 2016.

[2] Peter S Ozsváth, András I Stipsicz, and Zoltán Szabó. *Grid homology for knots and links*, volume 208. American Mathematical Soc., 2017.