Properties of $\widehat{GH}$ (§ 7.1 & 7.3)

Last week (Roelien): $\widehat{GH}$ encodes (categorifies) the Alexander polynomial:

$$\chi_{gr}\left(\widehat{GH}(K)\right) = \Delta_K(t).$$

Motto / Devies: $\widehat{GH}$ not only encodes the properties of Δ_K, but also strengthens them!

Today!

(1) Recall that for an oriented knot K, $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$, i.e., if

$$\Delta_K(t) = \sum_i a_i t^i, \text{ then } a_i = 0 \text{ if } i \gg 0 \text{ or } i \ll 0.$$

$\mathcal{GH}$ gets its counterpart:

<u>Proposition</u>: $\mathcal{GH}^-(K)$ is a finitely generated $\mathbb{F}[U]$-module. Likewise, $\widehat{\mathcal{GH}}(K)$ is a finite dimensional $\mathbb{F}$-vector space (in particular, every $\widehat{\mathcal{GH}}_d(K,s)$ is also finite dimensional)

<u>Proof</u>. $\mathcal{CG}^-(K) = \underbrace{\mathbb{F}[V_1 \dots V_n]}_{\text{Noetherian}} [\underbrace{S(g)}_{\text{finite}}]$ is finite generated $\Rightarrow \mathcal{GH}^-(K) = \dfrac{\ker \partial^-}{\operatorname{Im} \partial^-}$ also finite-generated as $\mathbb{F}[V_1 .. V_n]$-module, and also as $\mathbb{F}[U]$-mod since $V_i \cdot = V_j \cdot$ in $\mathcal{GH}^-(K)$. Now use the exact triangle

$$\begin{tikzcd}
\mathcal{GH}^-(K) \arrow[rr, "U\cdot"] & & \mathcal{GH}^-(K) \arrow[dl, "\pi"] \\
& \widehat{\mathcal{GH}}(K) \arrow[ul, "\delta"] &
\end{tikzcd}$$

to conclude. $\square$

(2) Let $-K$ and $\overline{K}$ denote the orientation reversal and the mirror image of an oriented knot K. Then

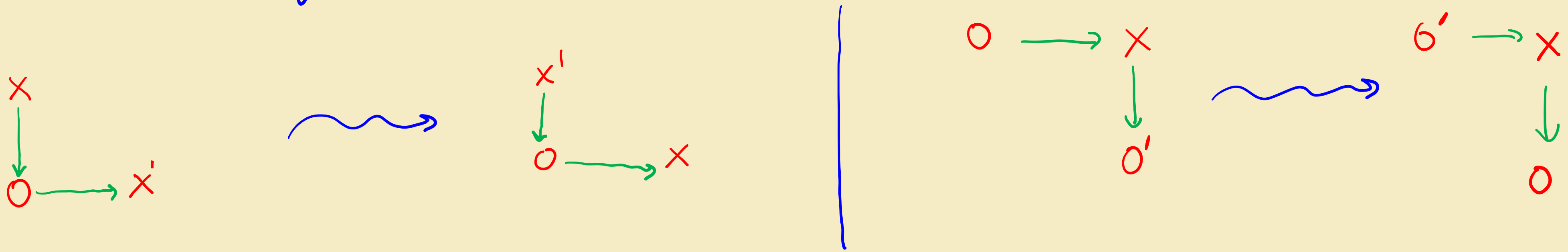
$$\Delta_{-K}(t) = \Delta_K(t) = \Delta_{\overline{K}}(t).$$

<u>Theorem</u>: 1) $\widehat{GH}(-K) \cong \widehat{GH}(K)$ as bigraded $\mathbb{F}$-vs (also true for GH^-).

2) $\widehat{GH}_d(\overline{K}, s) \cong \widehat{GH}_{2s-d}(K, s)$

Ie, $\widehat{GH}$ depends only on the isotopy type of the knot (regardless of the orientation) and it is sensitive under mirroring, unlike Alexander.

Proof. 1) If $\mathbb{G}$ is a grid diagram for K, then $-\mathbb{G} :=$ (grid diagram obtained by reflecting $\mathbb{G}$ across the diagonal) represents $-K$.



This induces a bijection $\phi : S(\mathbb{G}) \xrightarrow{\cong} S(-\mathbb{G})$. The key observation is that $\mathcal{J}(P,Q) = \mathcal{J}(\phi(P), \phi(Q))$, thus $M(x) = M(\phi(x))$ and $A(x) = A(\phi(x))$.

The reflection also induces a bijection $Rect^\circ(x,y) \xrightarrow{\cong} Rect^\circ(\phi(x), \phi(y))$, so ϕ extends to a chain isomorphism $\phi : CG^-(\mathbb{G}) \xrightarrow{\cong} CG^-(-\mathbb{G})$.

2) See 7.1.2.

$\square$

(3) The (normalised) Alexander polynomial is symmetric in t,

$$\underline{\Delta_K(t^{-1}) = \Delta_K(t)}, \quad \text{ie}, \quad \underline{a_i = a_{-i}}$$

We usually write it as $\Delta_K(t) = a_0 + \sum_{i>0} a_i(t^i + t^{-i})$.

<u>Proposition</u>. $\underline{\widehat{GH}_d(K,s) \cong \widehat{GH}_{d-2s}(K,-s)}$, for any K.

<u>Note</u>. This symmetry encodes (categorifies) the symmetry $a_i = a_{-i}$, since when taking Euler char.,

$$a_s = \sum_d (-1)^d \dim \widehat{GH}_d(K,s) = \sum_d (-1)^{d-2s} \dim \widehat{GH}_{d-2s}(K,-s) = a_{-s}$$

<u>Proof of the proposition</u>. Let $\mathcal{G}$ be a grid diagram for K and let $\mathcal{G}'$ be the diagram obtained by interchanging the roles of O's and X's. Then $\mathcal{G}'$ represents $-K$. Again there is a bijection $\phi : S(\mathcal{G}) \xrightarrow{\cong} S(\mathcal{G}')$, but this time it does <u>not</u> respect the bigrading. Let A, M for $\mathcal{G}$ and let $A' := \phi \circ A$, $M' := \phi \circ M$.

Since $A(x) = \frac{1}{2}\left[M_O(x) - M_X(x) - (n-1) \right]$, we get

$$M(x) - M'(x) = 2A(x) + n - 1$$

$$A(x) + A'(x) = 1 - n$$

which yields $\widetilde{GH}_d(\mathcal{G}, s) \cong \widetilde{GH}_{d-2s+1-n}(\mathcal{G}', -s+1-n)$. Now use a Poincaré polynomial argument together with the relation $\widetilde{GH} \cong \widehat{GH} \otimes W^{\otimes n-1}$ to conclude. $\square$

(4) Let K, K' be knots and let $K \# K'$ be its connected sum. A classical property of Alexander is that

$$\Delta_{K\#K'}(t) = \Delta_K(t) \cdot \Delta_{K'}(t).$$

$\widehat{GH}$ generalises this property:

<u>Proposition</u>: $\widehat{GH}(K \# K') \cong \widehat{GH}(K) \otimes_{\#} \widehat{GH}(K')$, ie

$$\widehat{GH}_d(K\#K', s) \cong \bigoplus_{\substack{d = d_1 + d_2 \\ s = s_1 + s_2}} \widehat{GH}_{d_1}(K, s_1) \otimes_{\mathbb{F}} \widehat{GH}_{d_2}(K', s_2).$$

5 Alexander gives a lower bound for the knot genus: if

$$\Delta_K(t) = a_0 + \sum_{s>0} \quad (t^s + t^{-s}), \text{ then}$$

$$g(K) \geq \deg \Delta_K(t) = \max \{ s : a_s \neq 0 \}$$

$\widehat{GH}$ detects the knot genus (!):

Theorem (Ozsváth-Szabó, 04): For any knot K,

$$g(K) = \max \{ s : \widehat{GH}_d(K,s) \neq 0 \text{ for some } d \}$$

Kevin's talk today: $\geq$.