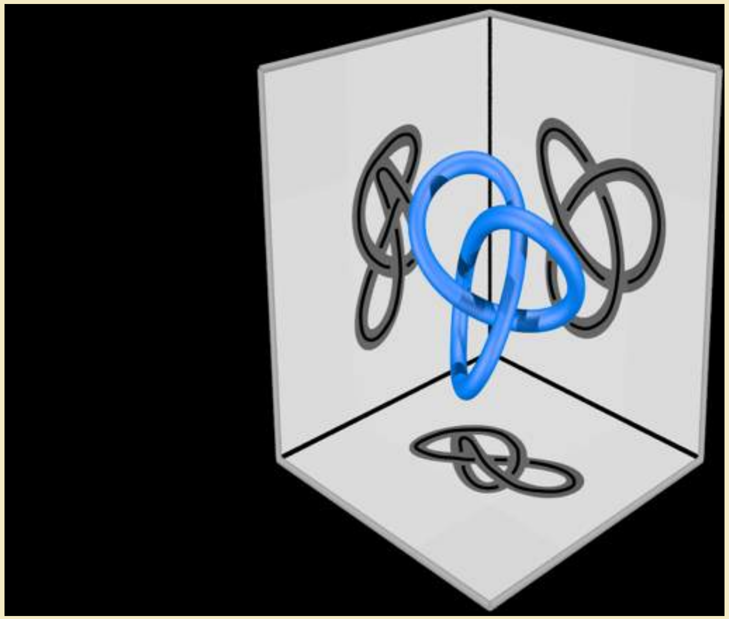
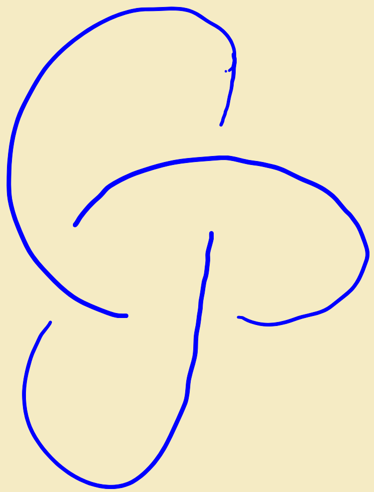
# Grid diagrams (§ 3.1)

A knot (ie, an embedding $S^1 \hookrightarrow \mathbb{R}^3$) is usually represented by a <u>knot diagram</u>, ie a planar diagram resulting from the projection $\mathbb{R}^3 \to \mathbb{R}^2$ with the information "over/under" at its crossings.



Knot in $\mathbb{R}^3$

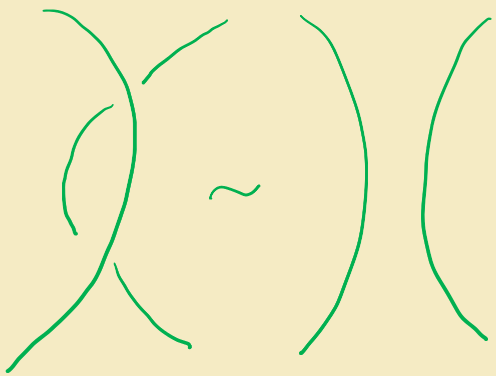


Knot diagram

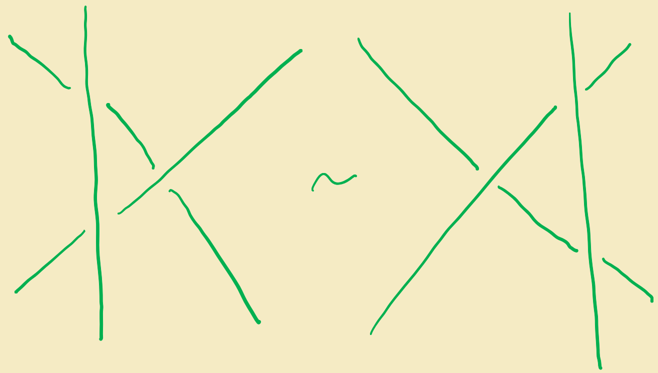
Since we are interested in studing knots (more generally, links) up to isotopy, one wonders what is the corresponding relation on knot diagrams.

<u>Theorem</u> (Reidemeister, 1926): Let $L, L'$ be links and let $D, D'$ be link diagrams representing them. Then $L, L'$ are isotopic if and only if $D, D'$ are related by a (finite) sequence of Reidemeister moves

R1

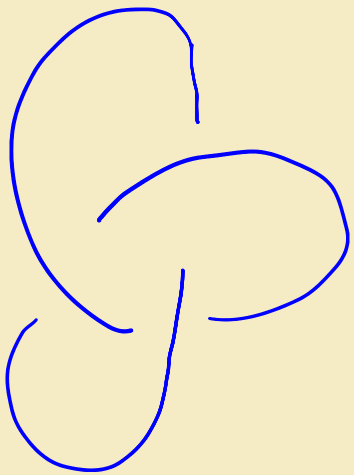


R2

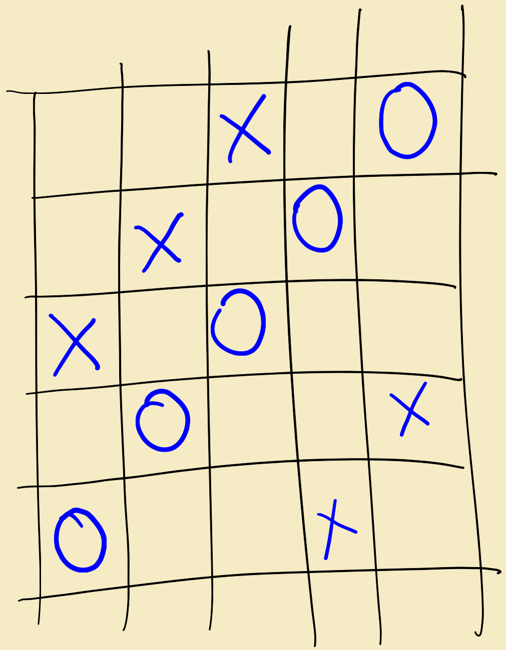


R3

The starting point of grid homology is to turn the topological data of the link into combinatorial data encoded in a grid diagram:

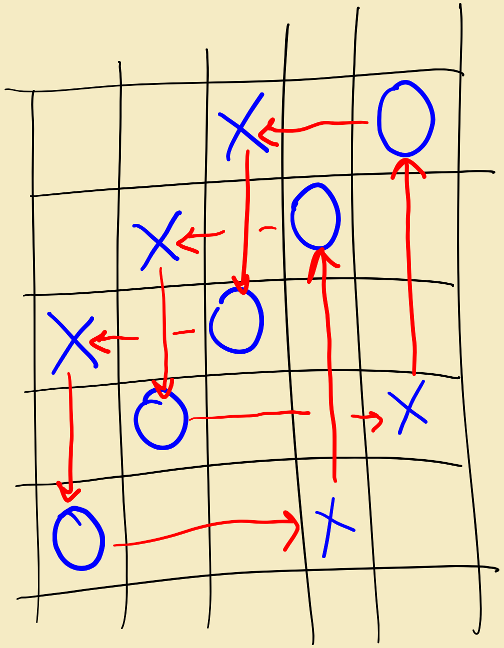


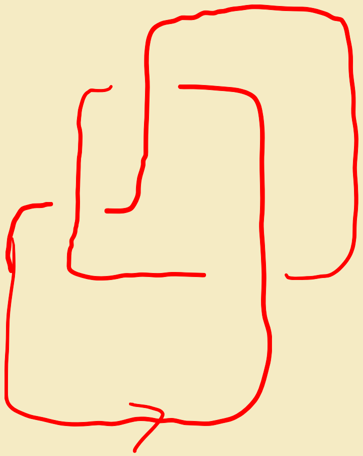
$\longleftrightarrow$



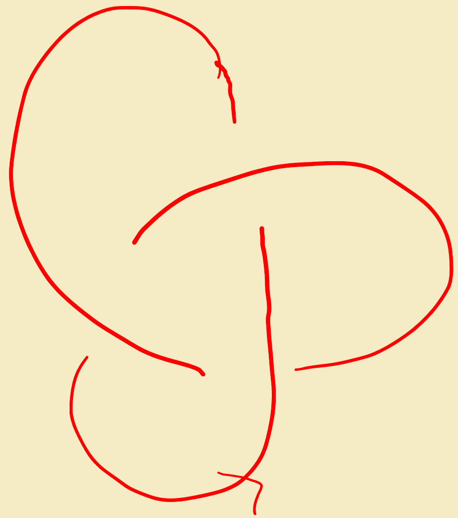
<u>Definition</u>: A (planar) <u>grid diagram</u> is a $n \times n$ grid together with $n$ labels $X$ and $n$ labels $O$, all in different positions, with the property that every column and every row has exactly one $X$ and one $O$.

* Every grid diagram gives rise to an oriented link diagram in the following way: draw oriented segments from $X$'s to $O$'s vertically, and from $O$'s to $X$'s horizontally. When the segments intersect, declare the vertical segment to be the over strand.



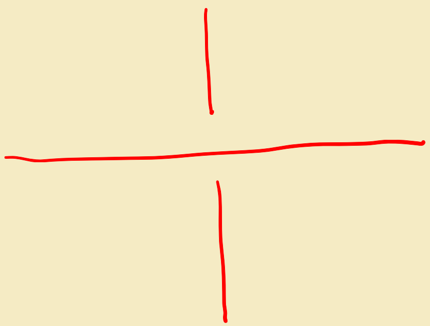


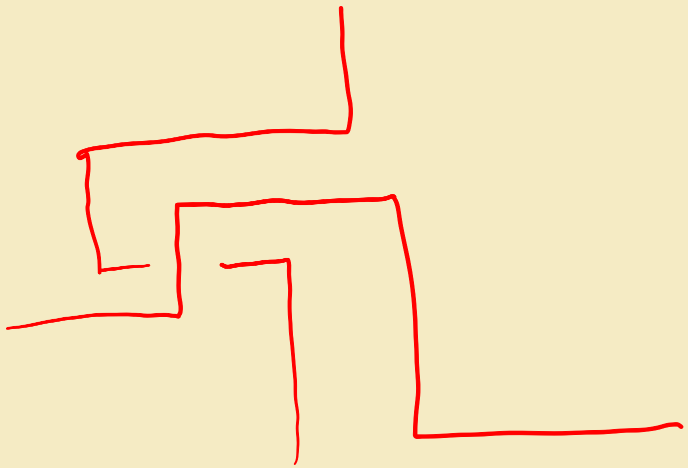
$\simeq$



<u>Lemma</u>: Every oriented link is represented by a grid diagram.

<u>Pf</u>. Arrange the strands of a link diagram so that it only has non-collinear horizontal and vertical strands. Now replace horizontal overcrossings as indicated:



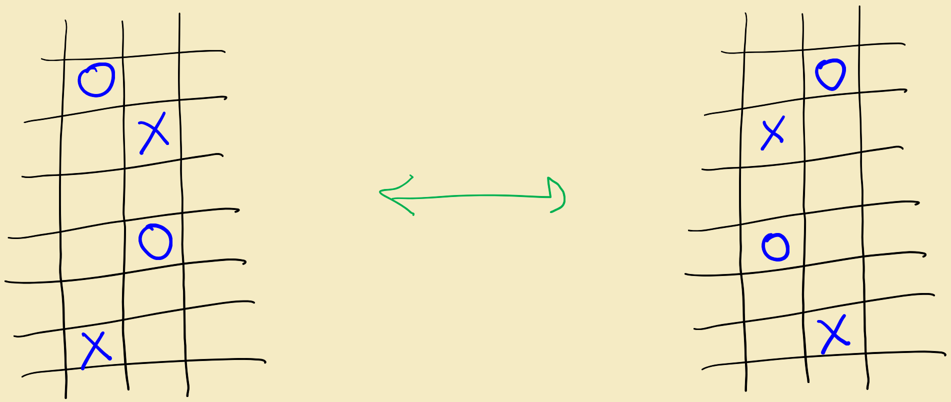


$\square$

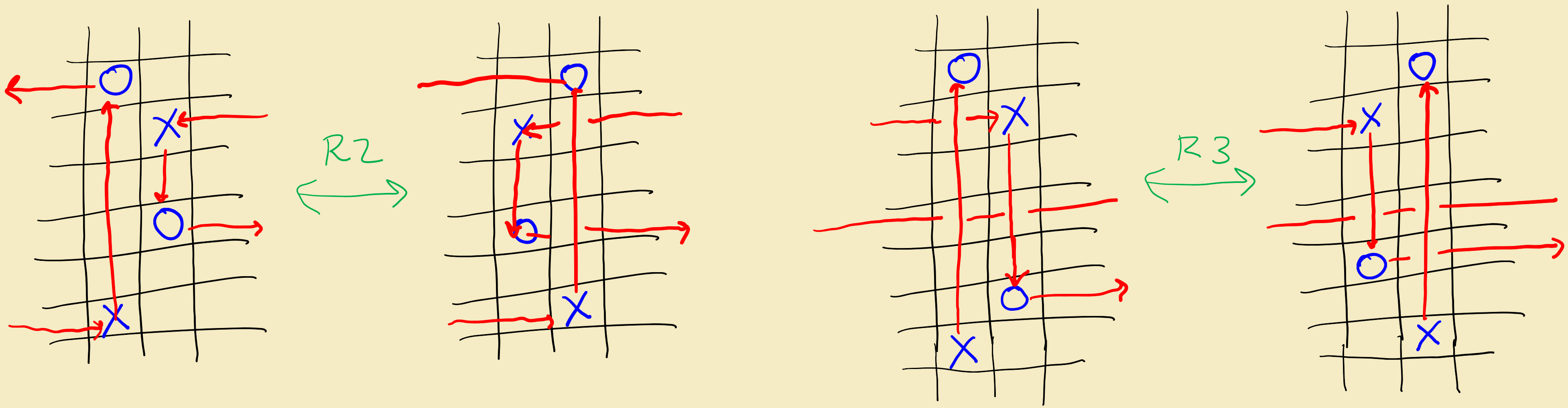
Q. When do two grid diagrams represent the same link?

Theorem (Cromwell, 95): Two grid diagrams represent the same link if and only if they are related by a (finite) sequence of the following transformations (called grid moves):

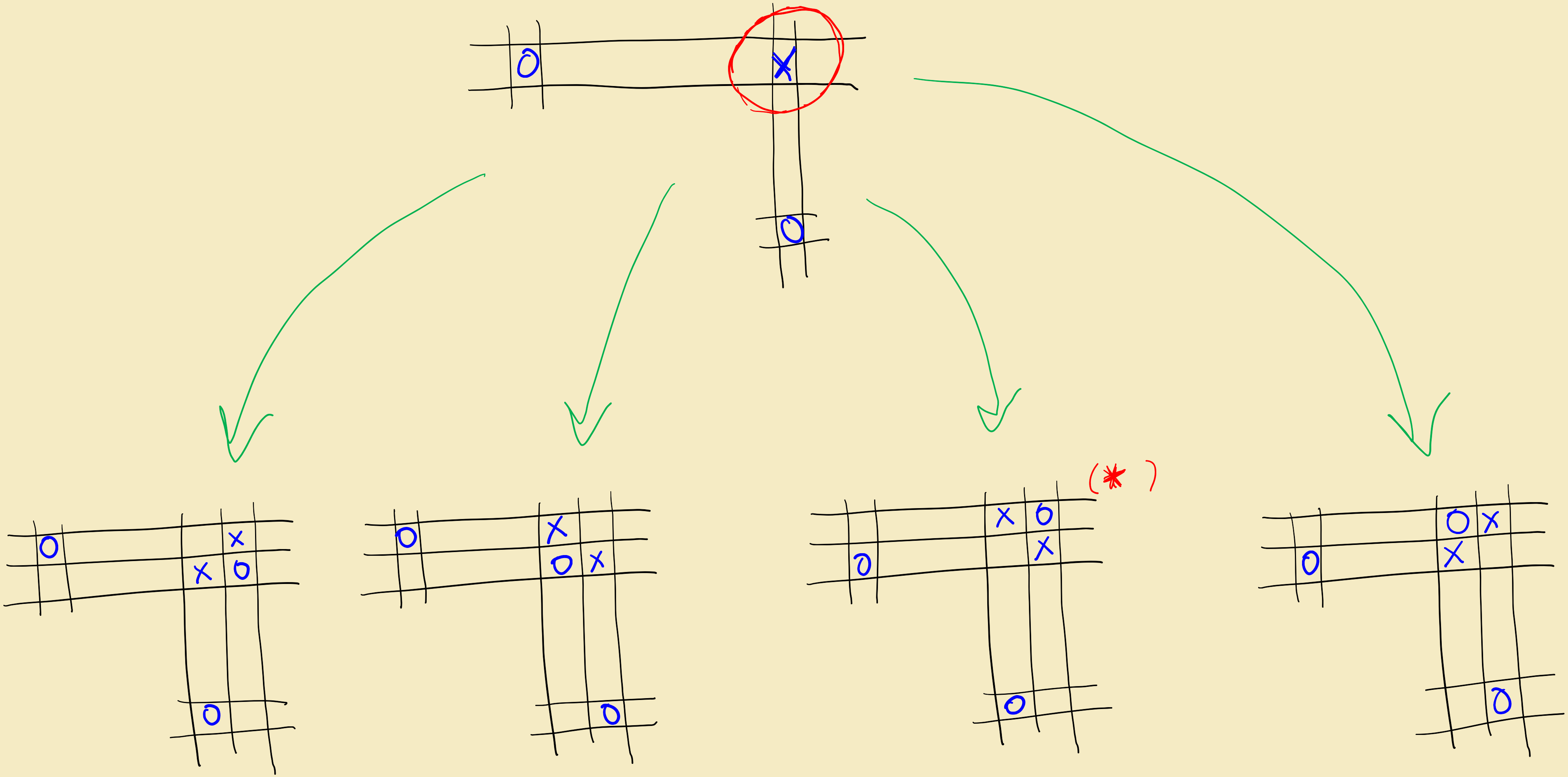
(1) Commutations: If two intervals (from X to O) associated to two consecutive columns (or rows) are either disjoint or one is contained in the other, then the two colums (or rows) can be interchanged.



This move corresponds to R2 & R3:




2) <u>Stabilisations</u>: Turn a $n \times n$ diagram into a $(n+1) \times (n+1)$ diagram in the following way: pick a marked square and remove the O/X in the same column, the O/X in the same row and the selected marked square itself. Now double the (empty) row and column giving rise to an $(n+1) \times (n+1)$ grid. There are four possible ways to insert markings in the new column and row. Each of them is a <u>stabilisation</u>.



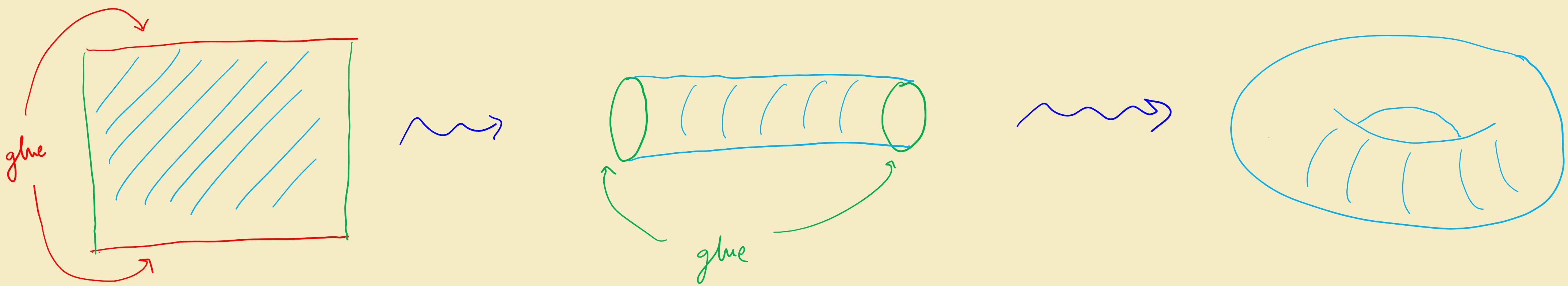
( this move corresponds to R1$^{(*)}$ & planar isotopies ).

3) <u>Destabilisations</u>: The inverse process of a stabilisation.

To sum up, we have seen that there are bijections

$$\frac{\{\text{links in } \mathbb{R}^3\}}{\text{isotopy}} \cong \frac{\{\text{link diagrams}\}}{\text{Reidemeister moves}} \cong \frac{\{\text{grid diagrams}\}}{\text{grid moves}}$$

- Toroidal diagrams : We will also make use of toroidal diagrams. First recall that the torus $S^1 \times S^1$ can be obtained by identifying opposite edges of $[0,1]^2$:



• If $\mathcal{G}$ is a planar grid diagram, a toroidal grid diagram is obtained by making the same identification on $\mathcal{G}$. Likewise, a toroidal diagram gives rise to $n^2$ different planar diagrams (planar realisations). Any two such are connected by a sequence of cyclic permutations of rows and columns.