Grid diagrams and the Alexander Polynomial

Topics in Topology

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In this talk we will see the relation between grid diagrams and the Alexander polynomial. We will give a link invariant for grid diagrams and show that it coincides with the Alexander polynomial. For the proof we will follow [1] quite closely.

1 Alexander Polynomial

In last weeks talk we have defined the Alexander polynomial for links. As a recap, recall the *(symmetrized) Alexander polynomial* for a link L with Seifert matrix S was given by:

$$\Delta_L(t) = \det\Big(t^{-1/2}S - t^{1/2}S^T\Big).$$

Another way of giving the Alexander polynomial is by means of the *skein relation*, which is given as:

Proposition 1. The **Alexander polynomial** $\Delta : \{\text{Links}\} \to \mathbb{Z}[t^{\pm 1/2}]$, $L \mapsto \Delta(L)$, of a link L, is defined by the skein relation:

1. $\Delta\left(\bigcirc\right) = 1$
2. $\Delta\left(\times\right) - \Delta\left(\times\right) = \left(t^{1/2} - t^{-1/2}\right)\Delta\left()(\right)$

Here $\bigcirc$ denotes the unknot.

The Alexander polynomial is an invariant of links, which means that given two equivalent (by means of Reidemeister moves and planar isotopy) links, we must have that they have the same Alexander polynomial. We will give a new link invariant based on grid diagrams which coincides with the Alexander polynomial.

2 Link invariant on grid matrices

Before we can prove anything, we need to discuss a few conventions. For a grid diagram, the markings X and O are placed in a grid. We say that the markings are at half integer coordinates, while the lattice points of the grid are at integer points. An example is given in figure 2. We also take the numbering of the lattice points from the bottom to the top and from left to right. This can also be seen in figure 2. Please notice these conventions depend on the author.

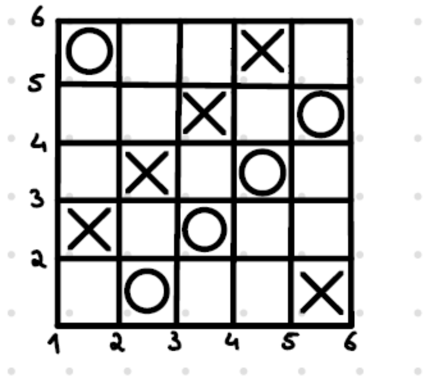


Figure 1: Convention for grid diagrams.

In order to be able to compute link invariants on a grid diagram we need to assign numbers to the diagram. This is done by means of *winding numbers*.

Definition 2.1. Let γ be a closed, piecewise linear, oriented curve in $\mathbb{R}^2$ and pick a point $p \in \mathbb{R}^2 \setminus \gamma$. The *winding number* $w_\gamma(p)$ of γ around p is defined as the algebraic intersection with the ray ρ from p to the point at infinity.

The winding number essentially measures how many times the curve goes around the point p.

A few remarks can be made about the winding number.

Remark. 1. In the case where γ is disconnected (i.e. a map going from a disjoint union of circles to $\mathbb{R}^2$) we say that the winding number is the sum of the algebraic intersections with the ray ρ. This is, for example, useful when considering links instead of knots.

2. The winding number might be known from complex analysis. Using complex analysis we can give define this winding number as the index of a curve $\gamma : I \to \mathbb{C} \cong \mathbb{R}^2$ at the point p. The index $\mathrm{ind}_p(x)$ is defined as:

$$\mathrm{ind}_p(x) = \frac{1}{2\pi i} \int \frac{dz}{z-p}.$$

3. In algebraic topology, we can see the winding number $w_\gamma(p)$ of $\gamma : S^1 \to \mathbb{R}^2 - \{p\}$ as the integer such that the induced map on the first homology group to $\gamma_* H_1(S^1; \mathbb{Z}) = \mathbb{Z} \to \mathbb{Z} = H_1(\mathbb{R}^2 - \{p\}; \mathbb{Z})$ is multiplication by $\omega_\gamma(p)$. Here we use that $\mathbb{R}^2 - \{p\}$ is homotopy equivalent to the circle. By the isomorphism between H_1 and the fundamental group π_1 we find that the same construction works if we replace H_1 with π_1.

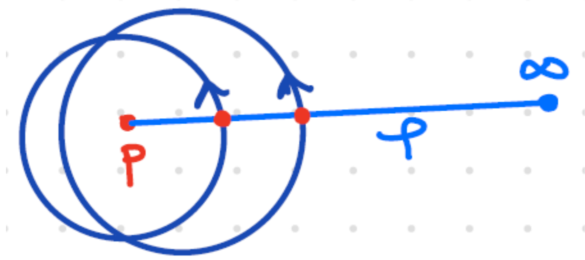


Figure 2: Winding number for a curve

The winding number of the figure 2 is +2, as the ray ρ intersects the curve γ twice in counter-clockwise direction. Here we will use the convention that the counter-clockwise direction is the positive direction.

For a lattice point in a grid diagram $\mathbb{G}$ we can find the winding number for p by making a ray ρ from p to the point at infinity along the horizontal axis. If the ray ρ intersects the lines between the markings while the lines go in a clockwise direction, then we add one to the winding number. If the ray ρ intersects the lines while in counter-clockwise motion, then we subtract one from the winding number. If the lattice point is outside all the markings we assign this point winding number zero. In figure 2 the winding numbers for a given grid diagram are given.

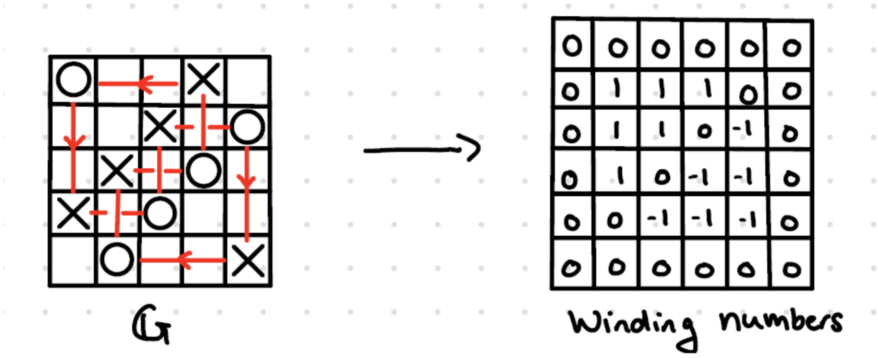


Figure 3: Winding numbers for a grid diagram

Now that we have winding numbers, we can define *grid matrices.*

Definition 2.2. For a given grid diagram $\mathbb{G}$, the *grid matrix* $M(\mathbb{G})$ is defined by setting the $(i,j)^{\text{th}}$ matrix element equal to t to the power the winding number of the lattice point $(j-1, n-i)$ in the grid diagram multiplied by -1. Or in formula:

$$(M(\mathbb{G}))_{i,j} = t^{-w((j-1,n-i))}.$$

Here we remark that the complicated index comes from our convention for the grid diagrams. Because of the index, we also leave out the top row and right-most column of lattice points.

As an example, consider the grid diagram $\mathbb{G}$ in figure 2. The grid matrix $M(\mathbb{G})$ is then given by:

$$M(\mathbb{G}) = \begin{pmatrix} 1 & t^{-1} & t^{-1} & t^{-1} & 1 \\ 1 & t^{-1} & t^{-1} & 1 & -1 \\ 1 & t^{-1} & 1 & t & t \\ 1 & 0 & t & t & t \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{♘}$$

To define a link invariant based on the grid diagram $\mathbb{G}$, the determinant of a grid matrix $M(\mathbb{G})$ is a good candidate. However, this 'invariant' would already fail on the unknot. This can be seen by computing the determinants of the following two grid diagrams, which both represent the unknot.

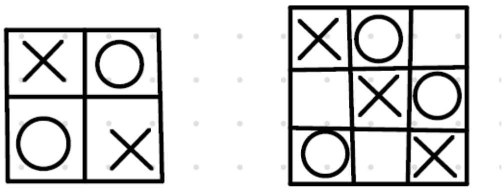


Figure 4: Two grid diagrams which determine the unknot

It turns out that if we normalize the determinant, we do find a link invariant. We will show in the next section that this normalized determinant even coincides with the symmetrized Alexander polynomial.

3 The main theorem

Before, we can state the main theorem we need to define two more ingredients. First of all, we define the quantity $a\,(\mathbb{G})$ for a given grid diagram $\mathbb{G}$. To find $a(\mathbb{G})$, we consider all the markings X and O in $\mathbb{G}$. For each, of these markings we look at the corners of the squares in which the marking is placed.The number $a(\mathbb{G})$ is then defined as the sum of all winding numbers of the marking corners, divided by eight.

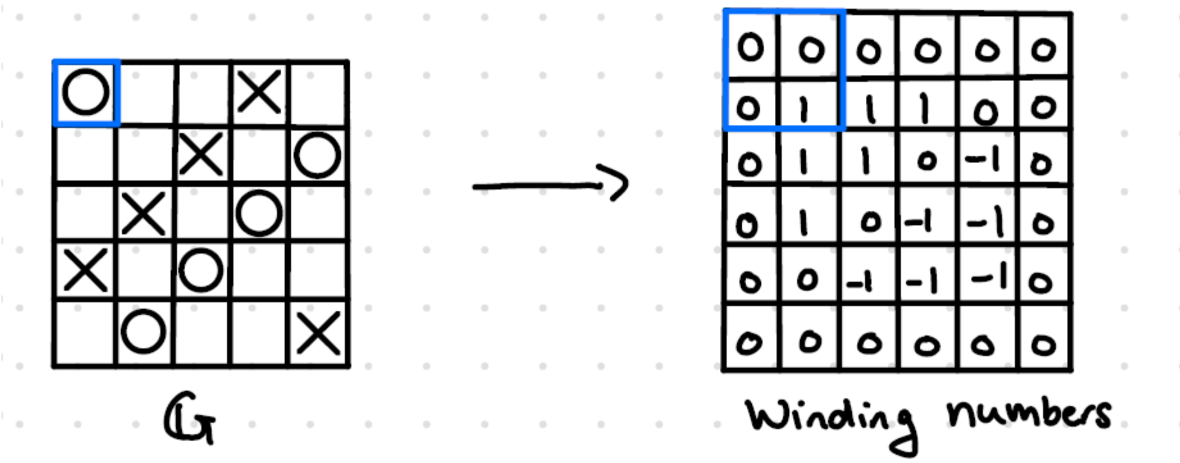


Figure 5: The quantity $a(\mathbb{G})$

If we look at the diagram $\mathbb{G}$ again, then for the blue square in figure 3 we see that the sum of the winding numbers of its corners is 1. If we compute all of the markings, we find that $a(\mathbb{G}) = 0$ in this case.

The last ingredient for the main theorem is the number $\varepsilon(\mathbb{G})$. This number is in the set $\{\pm 1\}$, and is defined as the sign of the permutation connecting $\sigma_{\mathbb{O}}$ and $(n, n-1, \ldots, 1)$. Here $(n, n-1, \ldots, 1)$ is defined by the permutation sending 1 to n, 2 to $n-1$ up until n to 1. Here $\sigma_{\mathbb{O}}$ is the list of vertical heights of the O markings.

In the example in figure 2 we find that $\sigma_{\mathbb{O}} = (5, 1, 2, 3, 4)$. This means we have a permutation:

$$1 \mapsto 5, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 3 \text{ and } 5 \mapsto 4.$$

We can now define the permutation τ such that $\tau \circ (n, n-1, \ldots, 1) = \sigma_{\mathbb{O}}$. This τ is in this case given by $(14)(23)$, which has sign $+1$.

We can now define a link invariant $D_{\mathbb{G}}(t)$ as follows:

$$D_{\mathbb{G}}(t) = \varepsilon(\mathbb{G}) \cdot \det(M(\mathbb{G})) \cdot (t^{1/2} - t^{-1/2})^{1-n} t^{a(\mathbb{G})}.$$

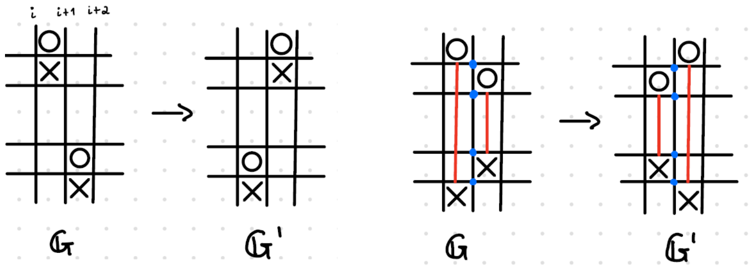
This definition seems to be picked out of thin air, however once we see its relation to the Alexander polynomial we see why this definition has been picked. This relation can be stated as the following theorem.

Theorem 3.1 (Theorem 3.3.6. in [1])**.** Let $\mathbb{G}$ be a grid diagram for a link L, then $D_{\mathbb{G}}(t)$ is a well-defined link invariant which coincides with the symmetrized Alexander polynomial $\Delta_L(t)$.

In order to prove this theorem, we need to show that $D_{\mathbb{G}}(t)$ is invariant under commutation and stabilization moves. Together with Cromwell's theorem, we then show that $D_{\mathbb{G}}(t)$ is a well-defined link invariant. Next, we will show that $D_{\mathbb{G}}(t)$ does satisfy this skein relation of the Alexander polynomial given in definition 1. We start at the beginning by the following lemma.

Lemma 1. The function $D_{\mathbb{G}}(t)$ is invariant under commutation moves.

Proof. We only sketch the proof of this lemma, the full proof can be found in [1, lemma 3.3.7.]. For stabilization moves, there are two cases, namely the case where the interval between the switched $X - O$ pairs are disjoint and the case where one interval is contained in the other. This is depicted in figure 3.



Let us first consider the case where the intervals are disjoint. Let us denote the original grid diagram by $\mathbb{G}$ and the grid diagram after the commutation move by $\mathbb{G}'$. The two swapped columns, are represented by three columns in the grid matrix. We notice that only in the middle one (denoted by $i+1$) changes. We notice that the absolute value of the elements do not change (this can be seen by computing the winding numbers), but because we swap two columns in the matrix, we do get an extra minus signs.

However, we notice that because we swap two columns the sign of $\varepsilon(\mathbb{G})$ changes as well. Because the winding numbers do not change, $a(\mathbb{G})$ does not change. And therefore, $D_{\mathbb{G}(t)}$ does not change as well.

For the case, where one interval is contained in the other (see figure 3), we should consider four sub cases. Namely, all permutations of positions of X and O in this diagram. For this case, we notice that the winding numbers of the lattice points in blue change. However, it can be shown that $a(\mathbb{G}) = a(\mathbb{G}') - 1$, so the extra t from the matrix, cancels with the lack of t from $a(\mathbb{G})$. The sign difference is dealt with as in the other case. $\square$

Lemma 2. The function $D_{\mathbb{G}}(t)$ is invariant under stabilization moves.

Proof. Again we only sketch the proof, the full proof can be found in [1, lemma 3.3.8.]. We again use a case distinction. We should consider all possible combinations of $X's$ and $O's$, but all can be proven by the same method. Let us consider the stabilization move $X : SW$ as depicted in figure 3. Let us consider the notation as in figure 3.

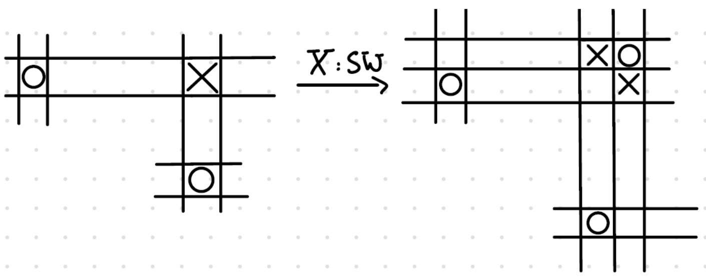


Figure 6: Stabilization proof

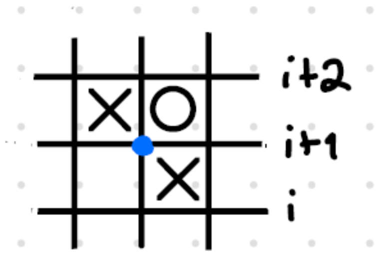
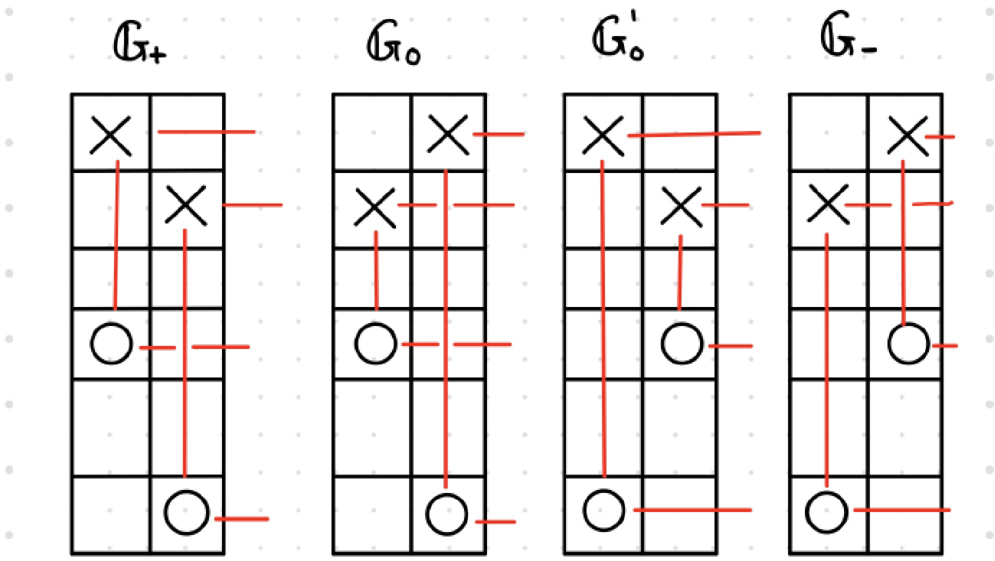


Figure 7: Conventions in the stabilization proof.

Because of the stabilization, we add a row and a column to our matrix. To see what the effect of this addition is for the determinant, we subtract row $i+1$ from $i+2$. What remains is one non-zero element (corresponding to the blue dot in figure 3). We can now compute the determinant of the minor obtained by removing the row and column corresponding to this non-zero element. We notice that this determinant is the same as the determinant of the original grid matrix. Changes in terms of other winding numbers and signs are as before in the previous lemma. □

Using the previous two lemma's and Cromwell's theorem we find that $D_{\mathbb{G}}(t)$ is indeed a well-defined link invariant. We can hence also write $D_L(t)$. It remains to be shown that $D_L(t)$ satisfies the skein relation. For this we need grid diagrams which represent the links ⤫, ⤬ and)(. It turns out, we can always make these, and they look like this:



With this terminology in place, we can show the following lemma.

Lemma 3. The invariant $D_L(t)$ satisfies the skein relation

$$D_{L_+}(t) - D_{L_-}(t) = (t^{1/2} - t^{-1/2})D_{L_0}(t).$$

Proof. This lemma can be found in [[1, lemma 3.3.11.]]. For this proof notice that $\mathbb{G}_+, \mathbb{G}_-, \mathbb{G}_0$ and $\mathbb{G}_0'$ only differ in two columns. Without loss of generality, we can assume these are the first two columns. If we compute the matrices of the grid diagrams, we can prove the following relation:

$$\det\left(\mathbb{G}_+\right) + \det\left(\mathbb{G}_-\right) = \det\left(\mathbb{G}_0\right) + \det\left(\mathbb{G}_0'\right).$$

Moreover, we can prove:

$$a(\mathbb{G}_-) = a(\mathbb{G}_+) = a(\mathbb{G}_0) + \frac{1}{2} = a(\mathbb{G}_0') - \frac{1}{2},$$

and we can show:

$$\varepsilon(G_+) = -\varepsilon(\mathbb{G}_-) = \varepsilon(\mathbb{G}_0) = -\epsilon(\mathbb{G}_0').$$

If we put this all together, we find that indeed:

$$D_{L_+}(t) - D_{L_-}(t) = (t^{1/2} - t^{-1/2})D_{L_0}(t)$$

holds. □

The previous three lemma's prove theorem 3.1.

4 Example

We have now seen that $D_{\mathbb{G}}(t)$ coincides with the symmetrized Alexander polynomial. We now could work out the trefoil knot example and see that $D_{\mathbb{G}}(t)$ indeed coincides with the Alexander polynomial of the trefoil. Using the knot atlas [2] we find that the Alexander polynomial of the trefoil knot K should be given by:

$$\Delta_K(t) = t + t^{-1} - 1.$$

We notice that most of the work for the trefoil has been done, as $\mathbb{G}$ in 2 is the grid diagram related to the trefoil knot. This means we find that $\epsilon(\mathbb{G}) = +1$ and $a\,(\mathbb{G}) = 0$ as was found before. Moreover, we already found the matrix of $\mathbb{G}$ in equation (♘). We only need to compute the determinant of this matrix. Using Mathematica, we find that this determinant is given by:

$$\det(M(\mathbb{G})) = \frac{\left(-t^3 + 2t^2 - 2t + 1\right)(t-1)^3}{t^3}.$$

Now, we can compute $D(\mathbb{G})(t)$ as follows:

$$\begin{aligned} D_{\mathbb{G}}(t) &= \varepsilon(\mathbb{G}) \cdot \det(M(\mathbb{G})) \cdot (t^{1/2} - t^{-1/2})^{1-n} t^{a(\mathbb{G})} \\ &= \frac{\left(-t^3 + 2t^2 - 2t + 1\right)(t-1)^3}{t^3} (t^{1/2} - t^{-1/2})^{-4} \\ &= t + t^{-1} - 1 \\ &= \Delta_G(t) \end{aligned}$$

which indeed proves that the invariant $D(\mathbb{G})(t)$ and the Alexander polynomial should coincide.

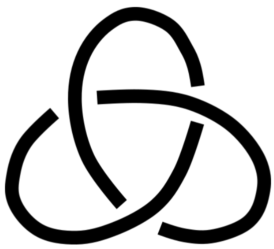


Figure 8: The trefoil knot

5 A few nice applications

In this section I will give a few (in my opinion nice) references related to the topic of this lecture. Of course we exclude here Grid homology as that will be the topic of the upcoming lectures.

5.1 Multivariable Alexander polynomial

The approach given here can be generalised to links and tangles. In this case the Alexander polynomial would not be enough and we need the *multivariable Alexander polynomial* which is defined on a collection of disjoint circles instead of just one circle. Using the same ingredients as above, as similar result holds for the Multivariable Alexander polynomial. For a link L with l components this is given as follows:

$$\Delta_G(t_1,\ldots,t_l) = \epsilon(G)\det M(\mathbb{G})\prod_{i=1}^{l}(1-t_i)^{-n_i}t^{a_i+\frac{n_i}{2}}.$$

More information on this result can be found in [3].

5.2 Quantum money

Knot theory and knot invariants have many applications. An unexpected application of grid matrices in general is the encryption of monetary systems. Using grid diagrams, it has been shown that a quantum money system can be made. In this case, a uniform superposition is generated on a grid diagram, giving a certain knot. Now certain knot invariants related to grid diagrams can be used as a key. More information on this application can be found in [4].

5.3 Software

There is a nice piece of software which generates the knot related to a given grid diagram. This can be found on `https://github.com/mwalczyk/grid-diagrams`.

6 Exercises

Exercise 1

Show that different planar realisations of a toroidal diagram represent isotopic links.

Exercise 2

Give two grid diagrams for the unknot such that the determinant of the corresponding grid matrices are not equal.

Exercise 3

Give a grid diagram for the figure eight knot and the hopf-link.

Exercise 4

Compute $D_{\mathbb{G}}(t)$ for the Hopf link and compare with its Alexander polynomial.

Exercise 5

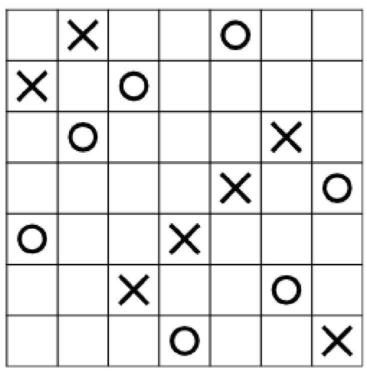
Exercise 3.1.5. from Grid Homology book [1].

Exercise 6

Show that a switch can be expressed as a sequence of commutations, stabilizations and destabilizations.

Exercise 7

Compute the winding numbers for the following grid diagram.



Exercise 8

Compute the Alexander polynomial for a general $(p, q)-$torus knot. Use Grid diagrams to do this.

References

[1] Peter S Ozsváth, András I Stipsicz, and Zoltán Szabó. *Grid homology for knots and links*, volume 208. American Mathematical Soc., 2017.

[2] Morrison S and Bar-Natan D. The knot atlas. `http://katlas.org/wiki/Main_Page`.

[3] Daniel Copeland and András Szucs. The multivariable alexander polynomial and thurston norm.

[4] Edward Farhi, David Gosset, Avinatan Hassidim, Andrew Lutomirski, and Peter Shor. *Quantum money from knots.* Proceedings of the 3rd Innovations in Theoretical Computer Science Conference, 2012.