Grid diagrams & the Alexander Polynomial

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Topics in Topology

Overview

1. Recap Alexander Polynomial
2. Main theorem and knot invariants
3. Winding numbers
4. Grid Matrices
5. Proof of the theorem
6. Extra things to think about

1

Recap of the Alexander Polynomial

Alexander Polynomial

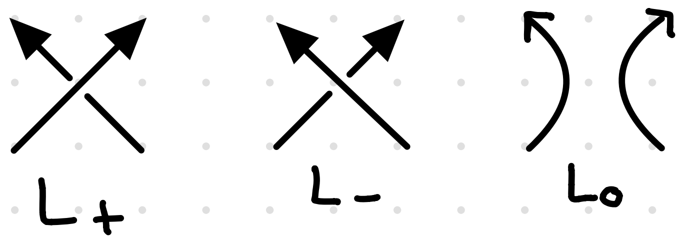
* knot invariant

Def. The Alexander Polynomial $\Delta_{\vec{L}}(t)$ of an oriented link $\vec{L}$ with Seifert matrix S is given by:

$$\boxed{\Delta_{\vec{L}}(t) = \det\left(t^{-1/2} S - t^{1/2} S^T\right)}$$

Alexander Polynomial via Skein relation

Skein triple:



Alexander Polynomial:

- $\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\,\Delta(L_0)$
- $\Delta(O) = 1$

2

Main theorem and Link invariants

Link invariants

Def. A Link invariant is a map:

$$f : \{\text{Links}\} \to S$$

set of links

↳ Some set: usually a field or Polynomial ring

s.t. if L_1 and L_2 are two equivalent links, we have $f(L_1) = f(L_2)$.

Rmk Not the other way around!

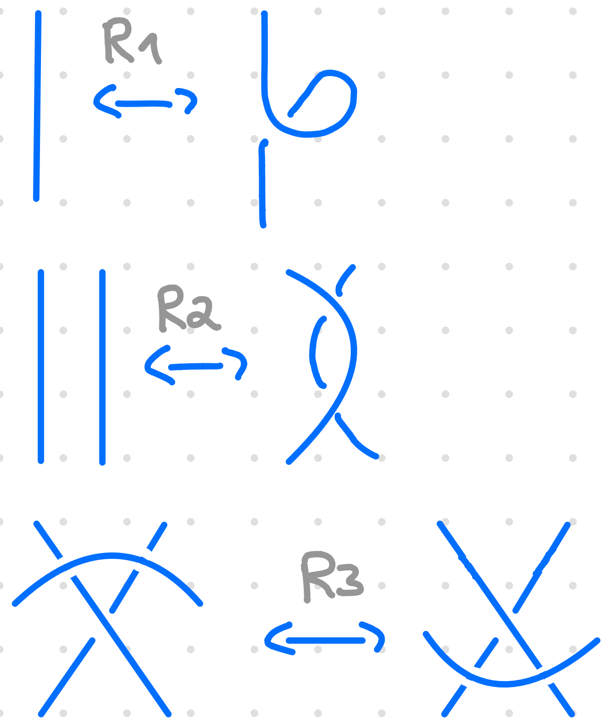
Well-defined Link invariant

We want a link invariant that does not

Change under:

1) Commutation
2) Stabilization
3) Planar isotopy

In general:



Main Theorem

Thm. [3.3.6]

Let $\mathbb{G}$ be a grid diagram for a link $\vec{L}$, then $D_{\mathbb{G}}(t)$ is a well-defined link invariant which coincides with the Symmetrized Alexander Polynomial $\Delta_{\vec{L}}(t)$.

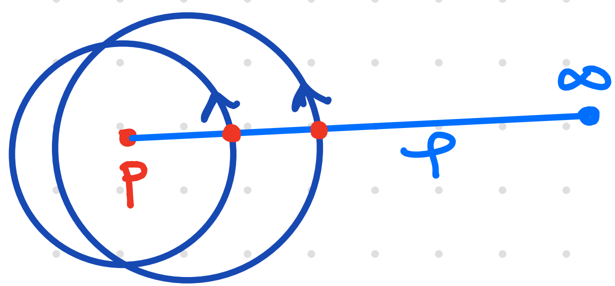
3

Winding numbers

Winding numbers

Def. Let γ be a closed, piecewise linear curve, oriented curve in the plane, and let p be a point in $\mathbb{R}^2 \setminus \gamma$.

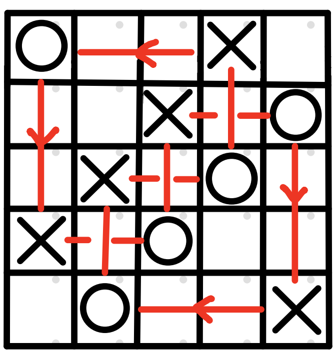
Given a ray φ from p to ∞, The Winding number $w_\gamma(p)$ is the algebraic intersection of φ with γ.



Winding number for Grid diagrams

The **winding number** for p can be found by making a ray φ from p to infinity, then :

1) -1 if φ intersects a clockwise grid line
2) $+1$ if —— // —— Counter-clockwise —— // ——.
3) 0 if p is outside closed circle.



$\longrightarrow$

0	0	0	0	0	0
0	1	1	1	0	0
0	1	1	0	-1	0
0	1	0	-1	-1	0
0	0	-1	-1	-1	0
0	0	0	0	0	0

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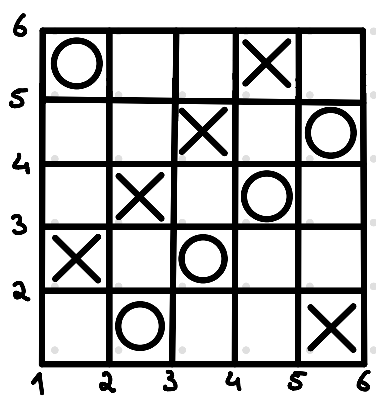
Grid Matrices

Convention

* Rows are labeled $1, \ldots, n$ from bottom to top.
* Columns are labeled $1, \ldots, n$ from left to right.



Grid Matrix

Def. For a grid diagram $\mathbb{G}$, the grid matrix $M(\mathbb{G})$ is defined, by setting

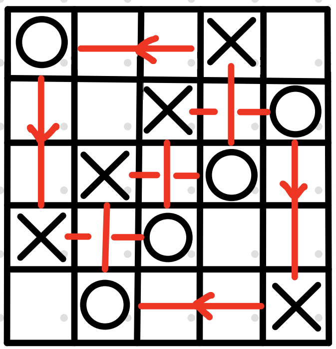
$$\boxed{(M(\mathbb{G}))_{i,j} := t^{-\omega((j-1,\, n-i))}}$$

Lattice point corresp. to matrix elt.

Rmk. * Weird index from convention.

* Leave out top row and rightmost column.

Example



$\mathbb{G}$

0	0	0	0	0	0
0	1	1	1	0	0
0	1	1	0	-1	0
0	1	0	-1	-1	0
0	0	-1	-1	-1	0
0	0	0	0	0	0

Winding numbers

$$M(\mathbb{G}) = \begin{pmatrix} 1 & t^{-1} & t^{-1} & t^{-1} & 1 \\ 1 & t^{-1} & t^{-1} & 1 & t \\ 1 & t^{-1} & 1 & t & t \\ 1 & 1 & t & t & t \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Determinant

The determinant is <u>not</u> a link invariant.

Check that

X	O
O	X

X	O	
	X	O
O		X

Have different determinants.

Two more terms

For every marking $\mathbb{X}$ and $\mathbb{O}$, compute the sum of the winding numbers of the corners of the square of the marking.

$a(\mathbb{G})$ is the sum of these winding numbers divided by 8, for all $\mathbb{X}, \mathbb{O}$ in $\mathbb{G}$.

Example $a(G)$

O			X	
		X		O
	X		O	
X		O		
	O			X

G

$\longrightarrow$

0	0	0	0	0	0
0	1	1	1	0	0
0	1	1	0	-1	0
0	1	0	-1	-1	0
0	0	-1	-1	-1	0
0	0	0	0	0	0

Winding numbers

$$a(G) = \frac{1+1+3-1+3-3+1-3-1-1}{8}$$

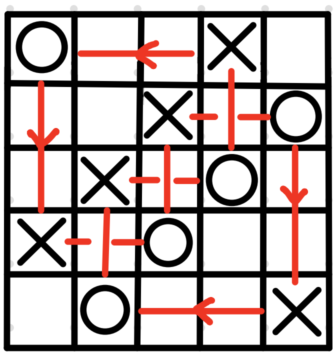
$$= 0$$

Two more terms (ctd.)

Let $\varepsilon(\mathbb{G}) \in \{\pm 1\}$ be the sign of the permutation connecting $\sigma_{\mathbb{O}}$ and $(n, n-1, \dots, 1)$

↳ y-positions of O's.

E.g.:



$\mathbb{G}$

$$\sigma_{\mathbb{O}} = (5, 1, 2, 3, 4)$$

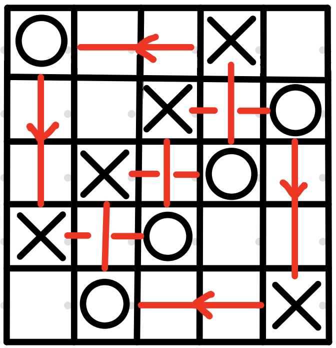
$$\varepsilon(\mathbb{G}) = 1$$

The Definition

For a grid diagram $\mathbb{G}$, define:

$$\boxed{D_{\mathbb{G}}(t) := \varepsilon(\mathbb{G}) \cdot \det(M(\mathbb{G})) \cdot (t^{1/2} - t^{-1/2})^{1-n}\, t^{a(\mathbb{G})}}$$

Example



$\mathbb{G}$

* $\det(M(\mathbb{G})) = (-x^3 - x^2 - x - 1)(-x+1)$
* $\varepsilon(\mathbb{G}) = 1$
* $a(\mathbb{G}) = 0$
* $n = 6$

Exercise: figure out which knot this is.

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Proof of the theorem

Main Theorem

Thm. [3.3.6]

Let $\mathbb{G}$ be a grid diagram for a link $\vec{L}$, then $D_{\mathbb{G}}(t)$ is a well-defined link invariant which coincides with the Symmetrized Alexander Polynomial $\Delta_{\vec{L}}(t)$.

What do we need to prove?

1) $D_G(t)$ is a well-defined Link invariant

   A) D_G is invariant under commutation moves.

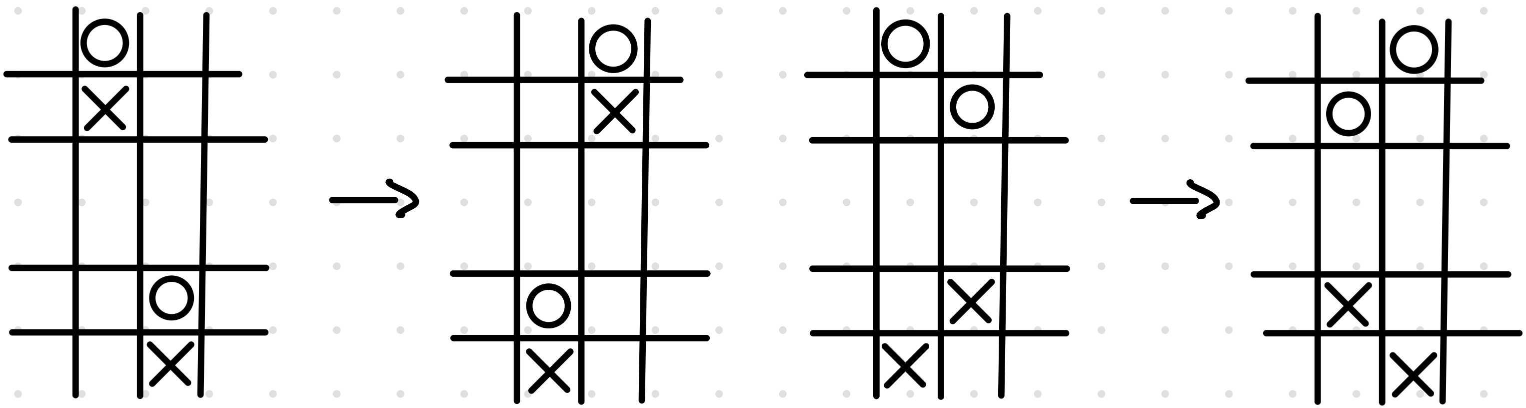
   B) D_G is invariant under stabilization.

2) $D_G(t)$ coincides with the Alexander Polynomial.

Invariance under commutation

Lemma [3.3.7]

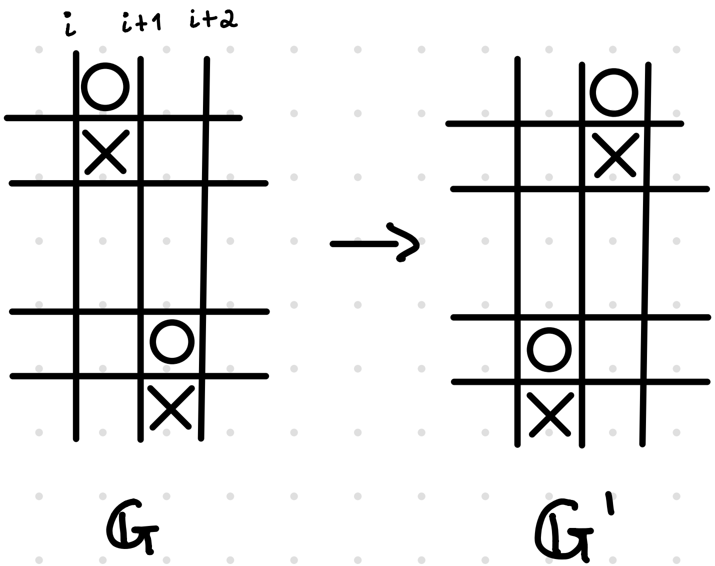
The function $D_G(t)$ is invariant under commutation moves.



NB: Same for row moves.

Sketch of the proof

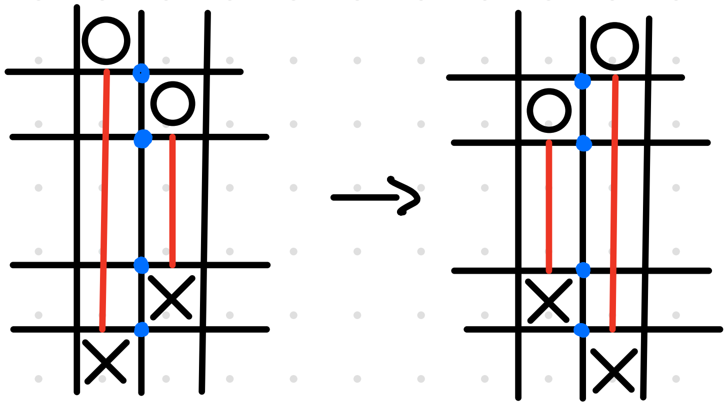
Case 1:



* Winding numbers do not change.
* Swap the two columns
* G and G' only differ in column $i+1$.
* Only a minus sign.
* $\mathcal{E}(G) = -\mathcal{E}(G')$.

Sketch of the proof (Ctd.)

Case 2:



$\mathbb{G}$ $\qquad\qquad\qquad$ $\mathbb{G}'$

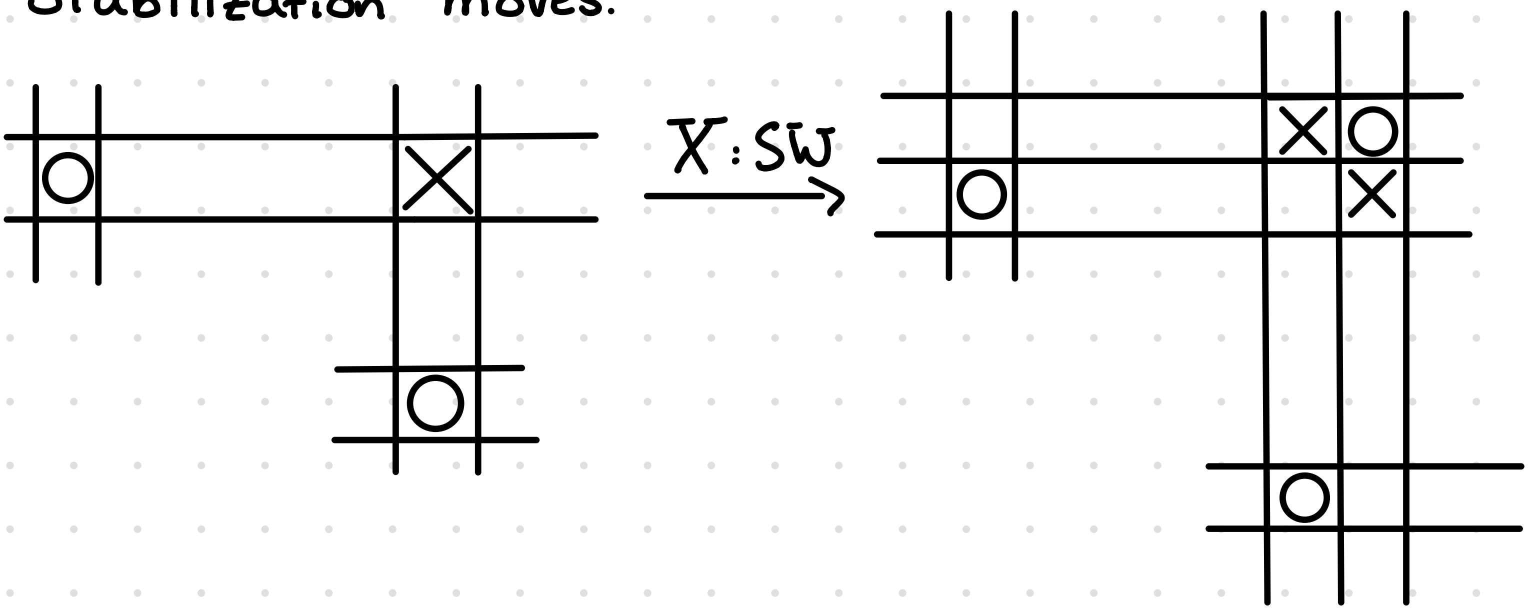
* Four subcases for position of X, O.
* Difference in winding number
* Sign difference
* $\varepsilon(\mathbb{G}) = -\varepsilon(\mathbb{G}')$
* $a(\mathbb{G}) = a(\mathbb{G}') - 1.$

$\square$

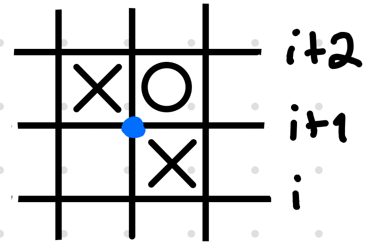
Invariance under stabilization

Lemma [3.3.8]

The function $D_G(t)$ is invariant under Stabilization moves.



Sketch of the proof



* X : SW
* Change row i+1
* Subtract i+2 from i+1
* Matrix has one non-zero term in this row.
* Determinant of minor = Determinant original
* The rest is similar to before. $\square$

$D_G(t)$ is a link invariant

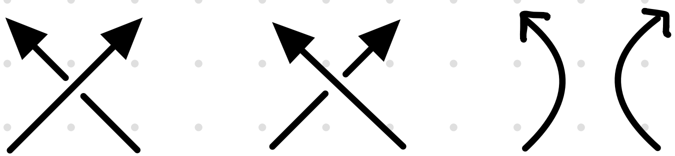
- Cromwell's theorem: Every grid diagram represents equivalent links iff links are related by a finite sequence of grid moves.
- We have shown: $D_G(t)$ is invariant under grid moves.
- So: $D_G(t)$ is a well-defined link invariant

For a link $\vec{L}$ we write: $D_{\vec{L}}(t)$.

Skein relation for grid diagrams

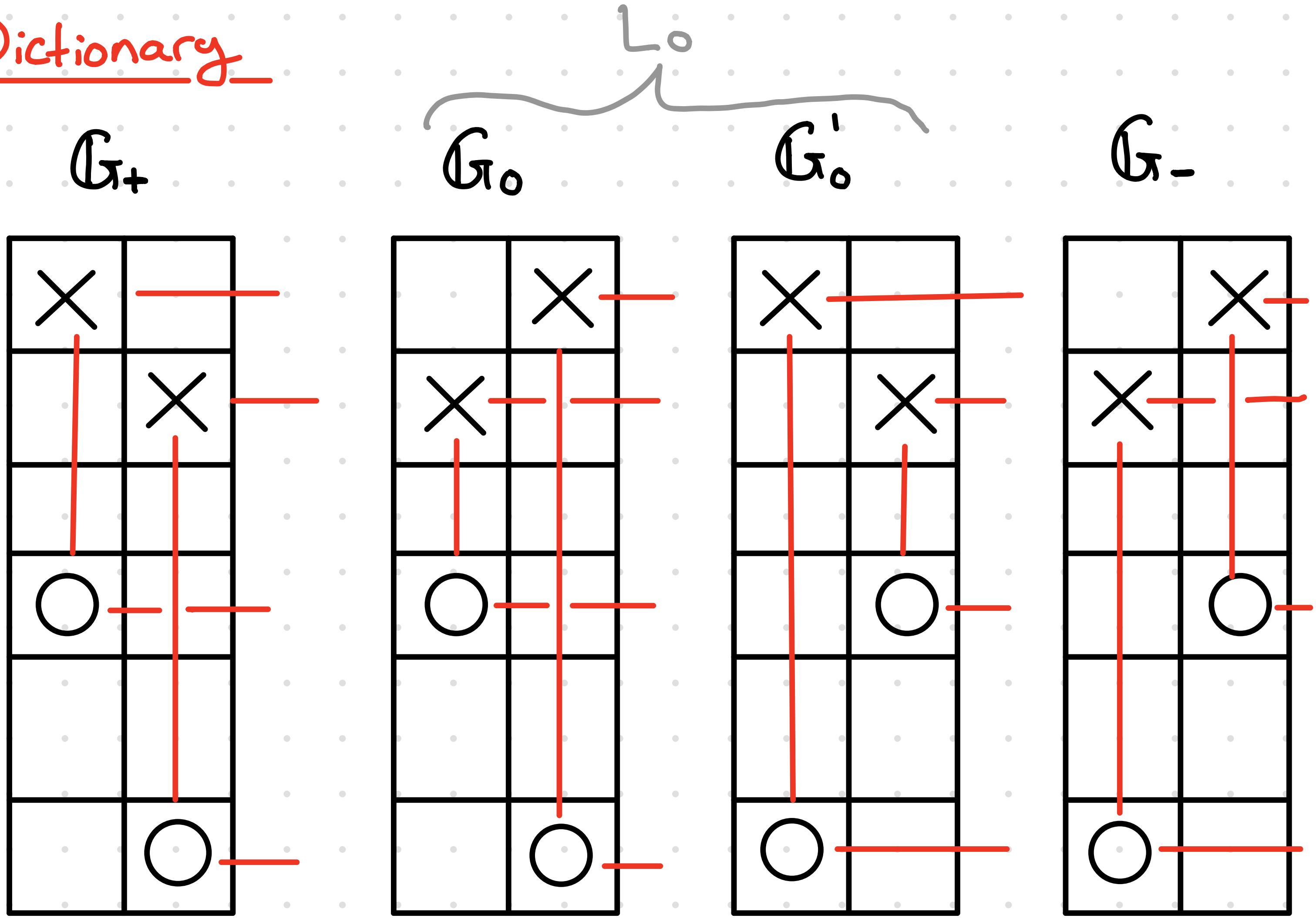
We will show $D_G(t)$ satisfies the skein relation.

We need grids for:



L_+ $\quad$ L_- $\quad$ L_0

Dictionary



Skein relation for grid diagrams

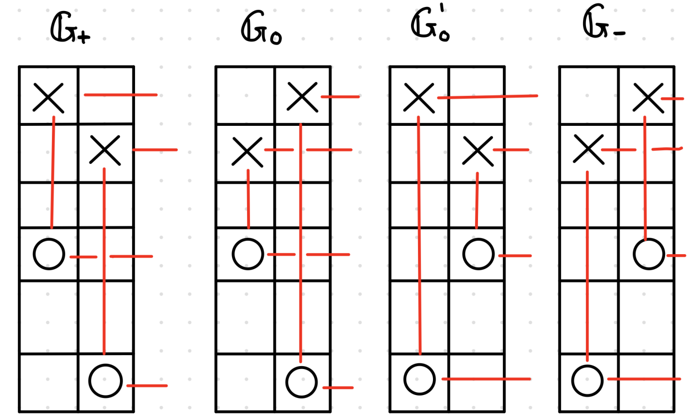
Lemma [3.3.11.]

The invariant $D_{\vec{L}}(t)$ satisfies the skein relation:

$$\boxed{D_{L_+}(t) - D_{L_-}(t) = (t^{1/2} - t^{-1/2})\, D_{\vec{L}_0}(t)}$$

Proof

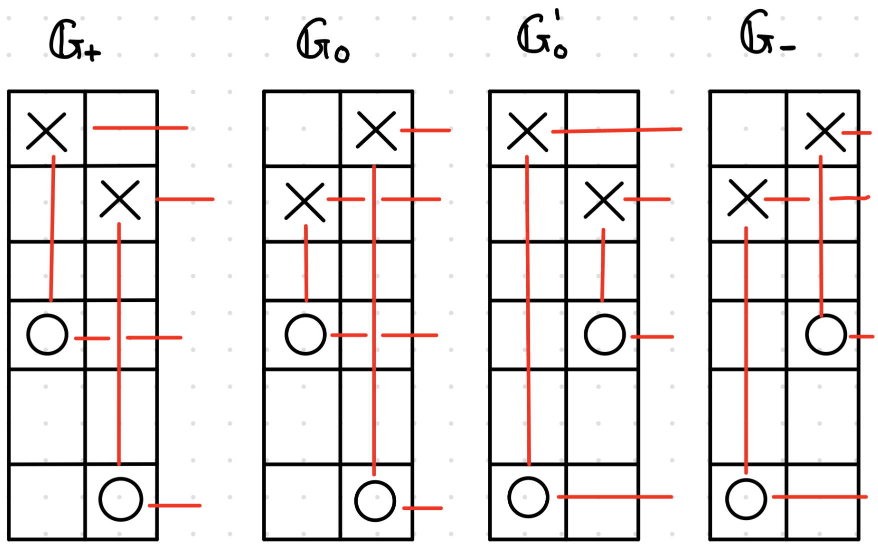
* G_+, G_0, G_-, G_0' differ only in two columns.



* w.l.o.g: Left-most two

Proof (ctd.)

Claim 1 $\det(M(G_+)) + \det(M(G_-)) = \det(M(G_0)) + \det(M(G_0'))$



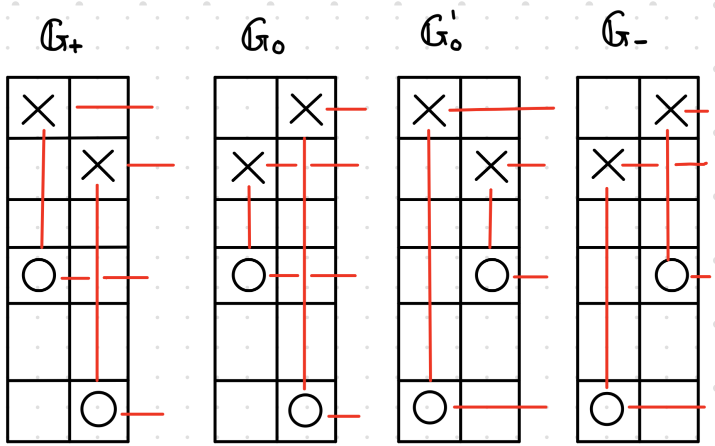
row operations and planar isotopy.

Proof (Ctd.)

Claim 2 $a(G_-) = a(G_+) = a(G_0) + \frac{1}{2}$

$$= a(G_0') - \frac{1}{2}$$

Claim 3 $\varepsilon(G_+) = -\varepsilon(G_-) = \varepsilon(G_0) = -\varepsilon(G_0')$



Proof (ctd.)

Combining the claims

$$D_{L_+}(t) - D_{L_-}(t) = \left(t^{1/2} - t^{-1/2}\right) D_{\vec{L}_0}(t)$$

$\square$

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A few nice things

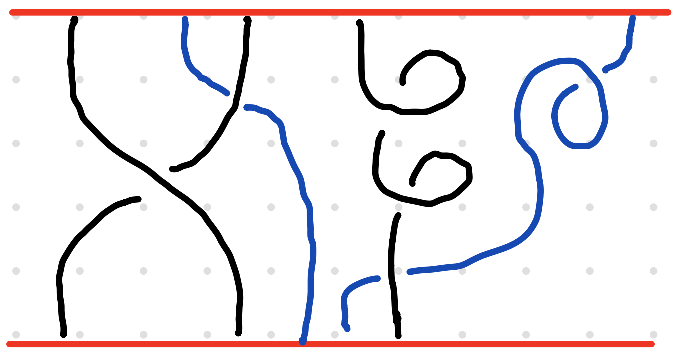
① Multivariable Alexander Polynomial

A similar result holds for the MVA

$$\boxed{\Delta_G(t_1,\dots,t_\ell) = \varepsilon(G)\det M(G)\prod_{i=1}^{\ell}(1-t_i)^{-n_i}\, t^{a_i+\frac{n_i}{2}}}$$

We have a winding matrix with winding numbers per link component.

Tangles



② Quantum Money

* Quantum money scheme based on this Alexander polynomial
* Quantum bit $\longrightarrow$ not forgeable
  $\longrightarrow$ Verification
* Uniform super position on a grid diagram $\Rightarrow$ knot
* Verification by Alexander polynomial

See notes for a reference

③ Nice vizualization program

Conclusion

1. Grid Matrices
2. Winding numbers
3. Grid diagrams and Alexander Polynomial.

For references see the notes