Heegaard Floer Homology

Topics in Topology

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outline

In these notes, an attempt is made to connect the original Heegard Floer homology theory to the grid homology discussed thusfar. For that end, we first take a deep detour through three dimensional manifold theory, wherefrom Oszvath and Szabo developed the Heegard Floer homology. The eventual subject should elucidate concepts and strategies that might have seemed *creatio ex nihilo* thusfar. With the benefit of hindsight, Szabo and Oszvath connected in the beginning of the 21st century the relatively novel homology of Andreas Floer to modern three manifold theory to great effect.

Heegaard splitting

This sections draws heavily from Saviliev [9], more detailed proofs can be found there.

Definition 1. *Def. a Heegard splitting of a 3-manifold M, is a union into two genus g solid handle-bodies (i.e. tori) H, H' along their boundary map $f : \partial H \to \partial H'$. We have equivalence $a \in H \sim b \in H'$ if $f(a) = b$,*

$$M = H \cup_f H' = H \sqcup H' / \sim$$

We restrict ourselves to orientation reversing maps.

> **Example**
>
> By gluing two solid genus 1 tori, $D^2 \times S^1$, along their boundary torus-shell $S^1 \times S^1$, you can get either $S^3 = \mathbb{R}^3 \cup \{\infty\}$ or $S^1 \times S^2$.
>
> The 3-sphere S^3 can be obtained by taking the complement of a solid g-torus.

Theorem 1. *Any (smooth, oriented) 3-manifold M has a Heegard splitting*

(Sketch.) Heegard splittings can be constructed out of the triangulation of the manifold. Replacing the 0-cells with small solid discs D^3, and the 1-cells (edges) with solid tubes $[-1, 1] \times D^2$, and joining these along their triangulation graph, we get a handlebody H. You can equivalently create it one fell swoop by taking a neighbourhood of the graph created by the 1-skeleton of your manifold.

For the 2 and 3-cells we do the same. They become tubes and solid 3-discs respectively, and union to another handlebody H' with same genus. After inflating both up so they intersect along a common boundary in the ambient space M, we have created a Heegaard splitting.

As shown by the example, splittings of any manifold can have arbitrary large genus. Adding a simple unknotted handle to H through ambient space M, creates a complementary handle in H' in a progress called *stabilisation*. The reverse, where a handle is removed, is called *destabilisation*. This allows us to consider if two splittings are *essentially* the same, i.e. they are homeomorphic after a finite series of stabilisations, destabilisations and splitting.

Theorem 2. *Any 3-manifold Heegaard splitting is essentially unique*

Proof. (Sketch, details [9, p.19-21]) It is a classical result of algebraic topology that any two triangulations of a manifold can be refined to a common descendant, a finer triangulation "contained" in both of the two original triangulations. The induced Heegaard splittings can be shown to be equivalent, they are stabilisations.

Above we have already discussed how a triangulation induces a Heegaard splitting, and by the triangulation common descendant in the previous paragraph, we only need to show any Heegaard splitting is related to the triangulation Heegaard splitting.

Assume $M = H \cup H'$. Both H, H' can be triangulated in a compatible manner, i.e. so that the triangulations are the 'same' along their common surface. This is done by first triangulating the *axial graph* of H, a graph homotopic to H itself interpreted as a 1-skeleton. This can be extended to a triangulation of H, then along the common boundary in a compatible manner to H'. Hence splitting $M = H \cup H'$ is essentially equivalent to a splitting from triangulation □

A powerful tool we will consistently employ is, under various form, referred to as Alexander's theorem. The form we use, is the following:

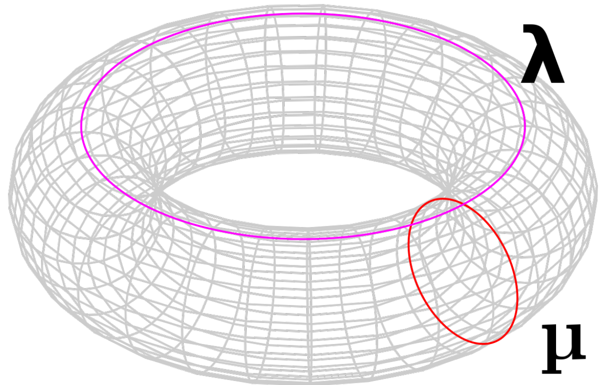
Theorem 3. *Any homeomorphism $f : D^3 \to D^3$ is entirely determined by f along the boundary $\partial D^3 = S^2$*

For details, see Saviliev for the form we use, or Hatcher [4] and [1] for other forms from the theorem can be concluded.

Gluing simple tori

Let us limit ourselves to gluing simple genus 1 handlebodies among their torus boundary. The 2-dim torus has the familiar fundamental group $\pi_1(T) = \mathbb{Z} < \mu, \lambda >$

(From Wikipedia)



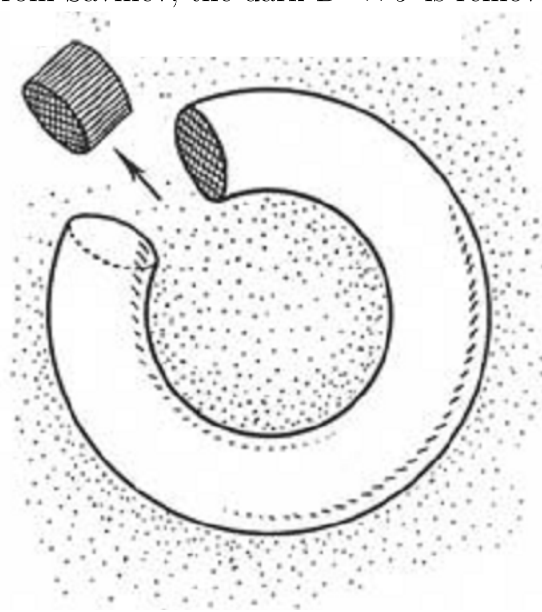
meridian μ, longitude λ.

To glue them together, we have a orientation reversing homeomorphism f on their respective boundaries, the torus T. This map descends to a map $f_\star$ on the fundamental group. We see that both the meridian and longitude have to be sent to linear combinations of the others' μ, λ. Due to non-degeneracy, they have to be linear independent.

Lemma 1. *The gluing $f : T \to T'$ along tori is completely determined by the image of the meridian μ.*

Proof. Snip the handle body along the thickened meridian $D^2 \times J$. The handle body $D^2 \times S^1$ decomposes as $D^2 \times J \cup D^3$.

(From Saviliev, the dark $D^2 \times J$ is removed)



But the boundary of $D^2 \times J \simeq D^3$ is S^2, as is the boundary of the complement. That means once we know how the boundary of this thickened meridian piece maps to the other torus, there is a unique way to attach the rest of the solid handle body by gluing in D^3. □

(Remark: This removing a solid torus from S^3 and gluing it back in along some meridian μ gives you *Lens spaces*. It is referred to as surgery along the unknot. A more complicated version of surgery along the link constructs *all* 3-manifolds as shown by Lickorish and Wallace in the 1960s.)

In a more complicated version you remove g copies of $J \times D^2$ in your g-handlebody. The linearly independent image of these g meridians among a genus g torus surface again uniquely determines how the complement ball D^3 is glued in.

Heegaard diagrams

Definition 2. *For a genus g surface Σ, we call a collection of closed simple curves $\{\alpha_1, \ldots \alpha_g\}$* attaching circles *if they are linearly independent in $H_1(Y)$. Equivalently $\Sigma \backslash \cup_i \alpha_i$ is connected. (You are asked to establish this in the exercises.)*

A *Heegaard diagram* is a genus g surface Σ, together with two attaching circles $\{\alpha\}, \{\beta\}$. Together they form the triple (Σ, α, β). The attaching circles determine unique handlebodies by the previous section, and how they are glued together.

For a knot $K \subset Y$, we have a *pointed Heegaard diagram* if the Heegaard diagram (Σ, α, β) induces a splitting of Y, and we have points $w, z \in \Sigma$, such that the knot K is the union of:

- the simple unknotted arc $w \to z$ in the 3-ball used in the attaching circles $\{\alpha\}$
- the simple unknotted arc $z \to w$ in the 3-ball used in the attaching circles $\{\beta\}$

Informally they 'avoid' the attaching circles in the surface, but for notational sake you draw the unknotted arc close to the surface Σ.

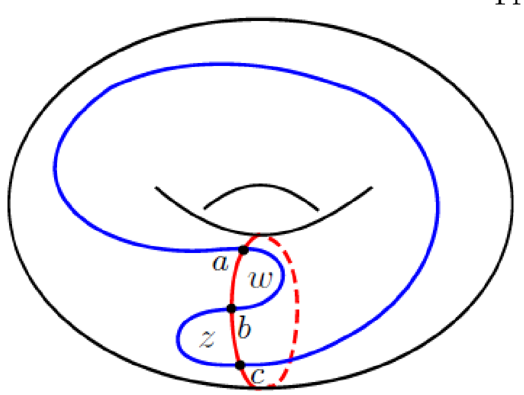
Definition 3. *This establishes the pointed Heegaard diagram $(\Sigma, \alpha\beta, w, z)$ of the knot $K \subset Y$.*

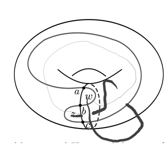
Example

As a example, this is the diagram of the trefoil knot.

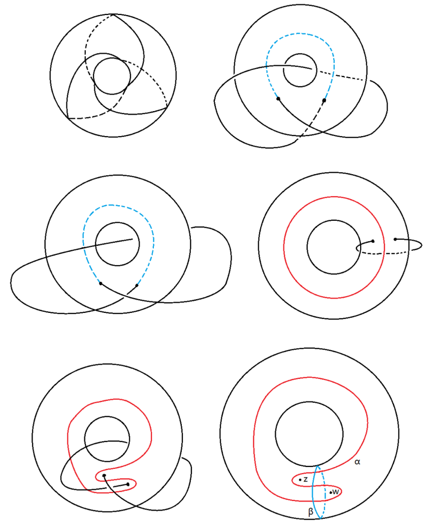
The arc $w \to z$ goes through the 'inside' of the solid torus, and from $z \to w$ it avoids the blue line by looping through the inside, and connecting from that end to w.

From Jennifer Hom, figure 5 [5]:





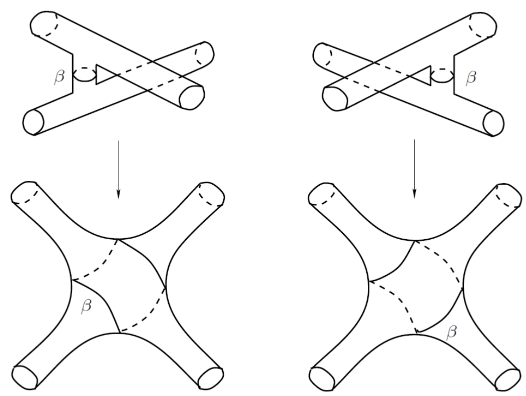
See also this picture from one of Stipsisz' students for the procedure detailed in [3] figure 1-4 for knots presentable in a genus $g = 1$ torus. The Goda et al example is a

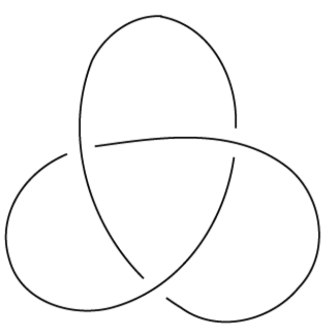


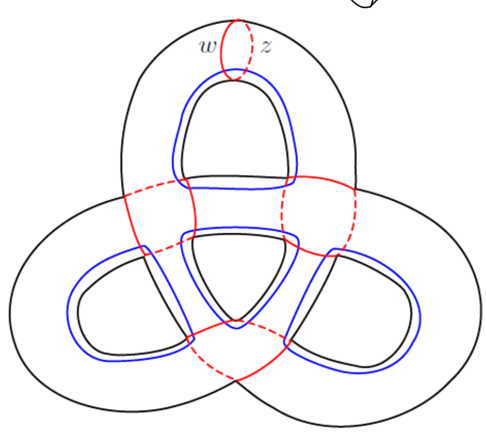
little too involved as initial example.

A more understandable algorithm is presented in 16.1.2 of the Grid Homology textbook [8]. Replace a Knot $K \subset S^3$ in general position w.r.t. a projection, with a thickened neighbourhood $U(K)$. This is a genus 1 handlebody. Connect the under/over crossings by a vertical (w.r.t. projection) solid tube, and a connecting loop β_i on each crossing. We have g as the sum of crossings plus one, but only $g-1$ loops β_i. We smash these crossings to a single cross-linked loop, see picture right.

Then connect g loops in $U(K)$ around the handlebody holes, giving us attaching circles α_g. Lastly at one point away from the crossing, we add w and z separated by the last attaching circle β_g. This gives a pointed Heegaard diagram for any Knot. The example due to Hom [5] of the trefoil is blow, with red the β, blue the α.







For doubly pointed Heegaard diagram $(\Sigma, \{\alpha\}, \{\beta\}, w, z)$ of genus g, define through the symmetric group S_g

$$\mathrm{Sym}^g(\Sigma) := \Sigma^{\times g}/\mathrm{S}_g$$

Lemma 2. *Sym*$^n(\Sigma)$ *is a smooth manifold for any* $n \in \mathbb{N}$

Proof is through the fundamental theorem of algebra, relating the unordered complex n-tuples in $\mathrm{Sym}^n(\Sigma)$ with the polynomial with those elements as zeroes, to the unique monic expression of the same polynomial. The factors describe a local homeomorphism to $\mathbb{C}^n$.

From hereon we work in a specific double pointed Heegaard diagram $(\Sigma, \{\alpha\}, \{\beta\}, w, z)$.

Associate to the attaching circles the following tori, as *subsets*[1] in $\mathrm{Sym}^g(\Sigma)$

$$\mathbb{T}_\alpha := \alpha_1 \times \cdots \times \alpha_g \qquad \mathbb{T}_\beta := \beta_1 \times \cdots \times \beta_g.$$

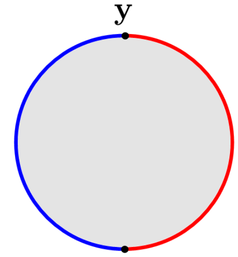
Definition 4. *Let S be the intersection of the tori*

$$S = \mathbb{T}_\alpha \cap \mathbb{T}_\beta = \cup_{\sigma \in S_g} \prod_{i=1}^{g} \alpha_{\sigma i} \beta_i$$

Note this is exactly how we obtained the states in chapter four of grid homology, each *state* corresponds to a permutation σ.

A *Whitney disc* is a map u from the unit disc $D^2 \subset \mathbb{C}$ to $\mathrm{Sym}^g\Sigma$ s.t.

- $\varphi(-i) = x$
- $\varphi(i) = y$
- $\varphi : \partial D^2 \cap \{\mathrm{Re} \geq 0\} \to \mathbb{T}_\alpha$
- $\varphi : \partial D^2 \cap \{\mathrm{Re} \leq 0\} \to \mathbb{T}_\beta$



i.e. one half of the outer circle is sent to $\mathbb{T}_\alpha$, the other to $\mathbb{T}_\beta$. See on the right, with red for $\mathbb{T}_\alpha$, and blue for $\mathbb{T}_\beta$. Picture from Hom [5].

Define $\pi_2(x, y)$ as the homotopy classes of Whitney discs from x to y.

(Remark: there is an equivalent picture of these 'discs' as strips, when they come from $\mathbb{R} \times i[-1, 1]$. They are equivalent but sometimes one insight is a little easier in a specific interpretation.)

Consider the Möbius transform τ that sends $i \to i$, $-i \to -i$ and 0 somewhere along the line $(-i, i)$. Pre-composing any Whitney disc u with τ, produces another Whitney disc $u \circ \tau$. This induces an $(-i, i) \simeq \mathbb{R}$-action on φ.

For any $\varphi \in \pi_2(x, y)$, let $\mathcal{M}(\varphi))$ be the collection of holomorphic[2] representatives of φ. By the above we have an action of $\mathbb{R}$, dividing this out we get the 'moduli' space

[1] we will continue to be sloppy with embedding non-symmetric spaces into the symmetrised space

[2] actually there is an almost complex structure, but this less general version is a little easier on the exposition

$$\widehat{\mathcal{M}}(\varphi) = \mathcal{M}(\varphi).$$

The maslov index $\mu : \pi_2(x, y) \to \mathbb{Z}$ measures the 'expected dimension' of the associated moduli space $\widehat{\mathcal{M}}$. The precise nature is outside this treatment, see Floer's original treatment [2].

Lemma 3. *For $g = 1$, $\widehat{\mathcal{M}}(\varphi)$ has one representative.*

The proof relies on the unique properties characterising the Möbius transforms on the unit disc in $\mathbb{C}$.

Define $n_z(\varphi) := \#\varphi^{-1}\{z\} \times \mathrm{Sym}^{g-1}\Sigma$.

Lemma 4. *The $n_z(\varphi)$ are all non-negative and finite for $z \in \Sigma \setminus (\alpha \cup \beta)$ [7, Lemma 3.2]*

Finally we arrive at the chain complex, which we will define over $\mathbb{F}_2[U, V]$ and with basis the intersections S. The differential is:

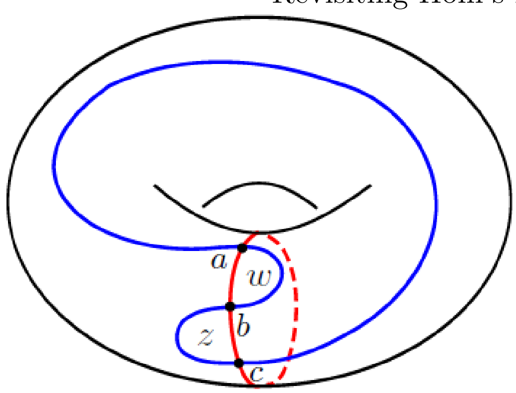
$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\varphi) U^{n_w(\varphi)} V^{n_z(\varphi)} y$$

Theorem 4. *$\widehat{\mathcal{M}}(\varphi)$ has finitely many elements for each power of U, V [7, prop. 2.15, remark 2.16, lemma 3.3]*

Theorem 5. *Restricting to $n_z(\varphi) = 0$, we get a chain complex with differential ∂^-. The homology hereof is equivalent to the unblocked grid complex homology.[8, theorem 16.4.1] [6]*

Example

Revisiting Hom's figure 5 for the trefoil[5]:



We get $\partial a = Ub$, $\partial b = 0$, $\partial c = Vb$.

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Exercise 1

Show $\{\gamma_i\}_{1\leq i\leq g}$ are attaching circles (lin. indep. disjoint curves in $H_1(\Sigma_g)$) for surface Σ if, and only if $\Sigma_g \setminus \cup_i \gamma_i$ is connected. (Hint: draw $4g$-polygon with edge and vertex identification.)

First try $g = 1, 2$ and then generalise.

Exercise 2

The figure 8 knot pointed diagrams:

a: with the first algorithm creating a large genus diagram

b: as genus 1 diagram by the trefoil-like procedure (resulting toroidal diagram is in 16.1 of book)

Exercise 3

Under assumption that the Möbius transforms from the unit disc to itself exist, are biholomorphic, and are determined by three points, show that:

All biholomorhpic functions on the unit disc $f : D^2 \subset \mathbb{C} \to D^2$ are Möbius transforms.

Exercise 4

Over field $\mathbb{Z}/2\mathbb{Z}$, finish the trefoil knot complex calculation by calculating the gradings of elements a, b, c, using $U = 1$, and $V = 1$.

Exercise 5

At the end of 16.1 Ozsvath et al give various genus 1 diagrams. Can you calculate the action of ∂ on each of their intersections?