Heegard Floer Homology

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Outline

Two parts:

- Heegard splitting
  - 3 Manifold theory
  - Heegaard diagrams
  - Examples
- Heegaard Floer Homology
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  - Chains
  - Example
- Kevin's problems?
- My problems?

80s: Floer developed his homology Lagrangian submfds of symplectic manifolds.

To prove Arnold's conjecture (Poincaré-Birkhoff generalisation) in some cases, intersection = fixed points [8]

00s: Ozsváth, Szabó, etc., create appropriate structures on 3 Manifolds to apply Floer Homology

3 Manifold $\implies$ Heegard Σurface $\implies$ tori Π_α, Π_β in $\mathrm{Sym}^{g}(\Sigma)$

Establishes Heegard Floer Theory of knots

Late 00s: combinatorial simplification $\implies$ grid homology

3 Manifold theory

Given two 3-mfds M_i with homeo $f : \partial M_1 \to \partial M_2$,

$$M := M_1 \cup_f M_2$$

If M_i same genus handlebodies, ∂M_i same, we have a **Heegard splitting**
Thm. All closed, orientable 3-mfd M have a Heegaard splitting.
Pf.

Triangulate M, tree inside 1=skeleton, skeleton 0, 1 cells, skeleton 2,3 cells handlebody

E.g. Sphere

Draw pictures of Sphere as $D^3 \cup D^3$, Σ_1 and Σ_g with stabilisation.

Thm. Heegards splitting are “essentially” unique
By “stabilisation”, isotopy, any splitting has a common descendant.

Pf Triangulations of mfd equivalent, which exist by above.Any Heegard is equivalent to triangulation decomp: axial graph of H_i, give triangulation to hole H_i, via boundary to H_i'.

sketch1

Alexander

Every 2-sphere in $\mathbb{R}^3$ bounds a 3-ball[3].

Every 2-sphere automorphism can be uniquely extended to entire ball (up to isotopy) [1].
Orientation preserving isotopic to identity

$\implies$ Any homeo $f : D^3 \to D^3$ determined by f on $\partial D^3 = S^2$
Sketch: $f : S^2 \to S^2$ on D^3 by $\hat{f}(r, \phi, \theta) = rf(\phi, \theta)$

Solid torus friends

Heegard splitting across g-torus,
$\pi_1(\text{Torus}) = \mathbb{Z} \oplus \mathbb{Z}$
draw Meridian

Lemma: image of μ completely determines gluing of two solid tori

$$D^2 \times S^1 = (D^2 \times \delta) \cup D^3$$

$\partial A \cup B = \partial A \cup B + a \cup \partial B$

60s Lickorish, Wallace: all 3-mfd by surgery on link in S^3

meridian $\mu \to \mu$ or λ

sketch2

Heegaard diagram

Σ genus g surface, then $\{\gamma_i\}_{1\leq i\leq g}$ are *attaching circles* if homologically linearly independent

Excercise: lin-indep $\iff$ $\Sigma\setminus\cup_i\gamma_i$ connected

Def. the **Heegaard diagram** is $(\Sigma,\{\alpha\},\{\beta\})$

prev page example attaching $T^1,\{\mu\},\{\lambda\}$

Knot Heegaard diagram

For knot K in 3-mfd Y, we have a **doubly pointed Heegaard diagram** $(\Sigma, \{\alpha\}, \{\beta\}, w, z)$ if:

- α, β attaching circles for $Y = (\Sigma, \alpha, \beta)$
- Arcs $w \to z$ and $z \to w$ via α resp. β 3mfds give rise to Knot

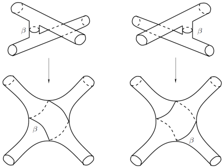
Recall 3-ball H_α attached along $\alpha_1 \dots \alpha_g$, let $w \to z$ in 3-ball
Likewise $z \to w$ in 3-ball H_β complement of $\beta_1 \dots \beta_g$

Practically runs near surface avoiding $\cup_i \alpha_i$ resp.$\cup_i \beta_i$
Unknot:

algorithm 1

Take knot K with knot diagram $\xrightarrow{1}$ thicken $U(K) \overset{2}{\to}$ connect crossings appropriately

$\overset{3}{\to}$ attaching circles α for genus holes, β for $g-1$ crossings, and β_g at w, z



algorithm 1+

Take knot K with knot diagram $\xrightarrow{1}$ thicken $U(K)$ $\xrightarrow{2}$ connect crossings appropriately
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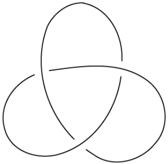
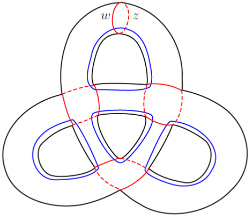
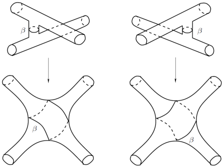
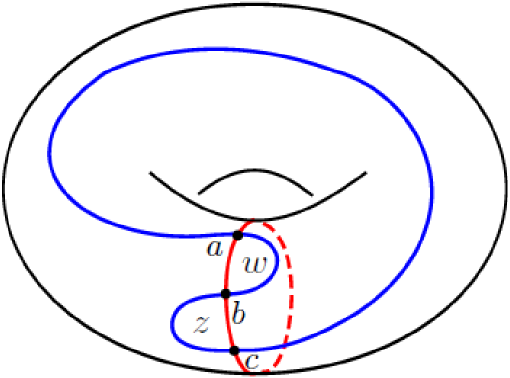
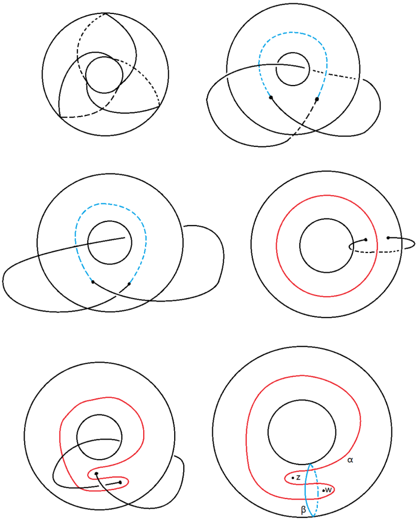






Trefoil alternative





α
z
w
β

sketch3

Riemann manifold

For doubly pointed Heegaard diagram $(\Sigma, \{\alpha\}, \{\beta\}, w, z)$, def

$$\mathrm{Sym}^g(\Sigma) = \Sigma^{\times g}/\mathrm{S}_g$$

Thm.: $\mathrm{Sym}^g(\Sigma)$ is a complex manifold
Sketch: $\Sigma \simeq \mathbb{C}$, locally $\simeq \mathbb{C}^g$

$$\begin{aligned} \mathrm{Sym}^g(\Sigma) \ni (w_1, \ldots, w_g) \leftrightarrow & \\ & (z - w_1) \cdots (z - w_g) = z^g + a_{g-1} z^{g-1} + \ldots a_0 \\ & \qquad \leftrightarrow (a_0, \ldots a_{g-1}) \in \mathbb{C}^g \end{aligned}$$

Whitney

$$\mathbb{T}_\alpha := \alpha_1 \times \cdots \times \alpha_g \; \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g \qquad \subset \mathrm{Sym}^g \Sigma$$

g-tori subspaces, assume intersect transversely, i.e.

$$S = \mathbb{T}_\alpha \cap \mathbb{T}_\beta$$

Whitney strips/disc $\pi_2(x, y)$ homotopy classes:
$x, y \in S$, a map $\varphi : D^1 \subset \mathbb{C} \to \mathrm{Sym}^g \Sigma$
$x \overset{\mathbb{T}_\beta}{\underset{\mathbb{T}_\alpha}{\Longleftrightarrow}} y$

$\widehat{\mathcal{M}}(\varphi) =$ all holomorphic representatives of φ modulo translation
Maslov

g=1 Maslov calculation

Fact 1: Möbius transforms on R-S $\overline{\mathbb{C}} = S^2$ completely determined by three points.

2: Only biholomorphic maps on unit disc are Möbius (exc. 3)

Lemma: $\widehat{\mathcal{M}}(\varphi)$ has one representative

Domain

$\Sigma \backslash \alpha \cup \beta = \cup_i \mathcal{D}_i^\circ$
$\partial \mathcal{D}_i \subset \mathbb{T}_\alpha \cup \mathbb{T}_\beta$
$\partial \partial_\alpha$

Heegaard-Floer chain

Def $n_z(\varphi) := \#\varphi^{-1}\{z\} \times \mathrm{Sym}^{g-1}\Sigma$

$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\varphi) U^{n_w(\varphi)} V^{n_z(\varphi)} y$$

$$\begin{aligned} \text{bigrading: } M(x) - M(y) &= \mu(\varphi) - 2n_w(\varphi) \\ A(x) - A(y) &= n_z(\varphi) - n_w\varphi \end{aligned}$$

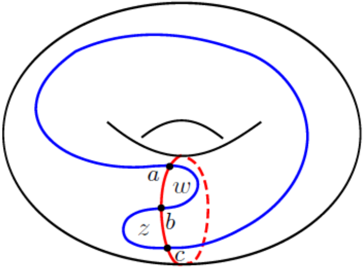
$n_z(\varphi) = 0 \implies \partial^-$
$\& n_w(\varphi) = 0 \implies \hat{\partial}$

Thm. (16.4.1) H-F homology ∂^- is equivalent to ∂^- grid homology
Lemma: φ is a product of rectangles

Trefoil example

$n_z(\varphi) := \#\{z\} \times \mathrm{Sym}^{g-1}\Sigma$

$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\varphi) U^{n_w(\varphi)} V^{n_z(\varphi)} y$$



$$H_\star(\text{trefoil}) = \mathbb{F}[aV + bU]_{(0,0)} \oplus \mathbb{F}_{(1,1)}[b]$$

Show Meridian matters, give diagram definition, pointed heegaard mfd. splitting examples

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exercise 1

Show $\{\gamma_i\}_{1\leq i\leq g}$ are attaching circles (lin. indep. disjoint curves in $H_1(\Sigma_g)$) for surface Σ if, and only if $\Sigma_g \setminus \cup_i \gamma_i$ is connected. (Hint: draw $4g$-polygon with edge and vertex identification.)

First try $g = 1, 2$ and then generalise.

exercise 2

The figure 8 knot pointed diagrams:

a: with the first algorithm creating a large genus diagram

b: as genus 1 diagram by the trefoil-like procedure (resulting toroidal diagram is in 16.1 of book)

exercise 3

Under assumption that the Möbius transforms from the unit disc to itself exist, are biholomorphic, and are determined by three points, show that: All biholomirhpic functions on the unit disc $f : D^2 \subset \mathbb{C} \to D^2$ are Möbius transforms.

exercise 4

Over field $\mathbb{Z}/2\mathbb{Z}$, finish the trefoil knot complex calculation by calculating the gradings of elements a, b, c, using $U = 1$, and $V = 1$.