# Gradings and a TQFT

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# Pre-additive category

**Definition**

A **pre-additive category** $\mathcal{C}$ is a category $\mathcal{C}$ such that for all $C, D, E \in \mathrm{Ob}(\mathcal{C})$ and for all $f, g \in \mathrm{Mor}(C, D)$ and $h, j \in \mathrm{Mor}(D, E)$, we have that

1. $\mathrm{Mor}(C, D)$ is an abelian group;
2. $h \circ (f +_{G_1} g) = (h \circ f) +_{G_3} (h \circ g)$;
3. $(h +_{G_2} j) \circ f = (h \circ f) +_{G_3} (j \circ f)$

for $G_1 := \mathrm{Mor}(C, D)$, $G_2 := \mathrm{Mor}(D, E)$ and $G_3 := \mathrm{Mor}(C, E)$.

- Extend (non-pre-additive) categories to pre-additive categories by allowing $\mathbb{Z}$-linear combinations of the ”original” morphisms.

# Mat($\mathcal{C}$)

## Definition

Let $\mathcal{C}$ be pre-additive. Then Mat($\mathcal{C}$) is a category with

1. Objects: direct sums $\bigoplus_{i=1}^{n} \mathcal{O}_i$ of objects $\mathcal{O}_i$ of $\mathcal{C}$;
2. Morphisms: a morphism $f$ from $\bigoplus_{i=1}^{n} \mathcal{O}_i$ to $\bigoplus_{j=1}^{n'} \mathcal{O}'_j$ is a matrix consisting of entries $f_{ij} : \mathcal{O}_i \to \mathcal{O}_j$;
3. if $f, \tilde{f} \in \text{Mor}(\bigoplus_{i=1}^{n} \mathcal{O}_i, \bigoplus_{j=1}^{n'} \mathcal{O}'_j)$ and $g \in \text{Mor}(\bigoplus_{j=1}^{n'} \mathcal{O}'_j, \bigoplus_{k=1}^{n''} \mathcal{O}''_k)$, then $f + \tilde{f}$ is given by $(f + \tilde{f})_{ij} = f_{ij} + \tilde{f}_{ij}$ and $g \circ f$ is given by

$$(g \circ f)_{ik} = \sum_{j=1}^{n'} g_{jk} \circ f_{ij}. \tag{1}$$

# Cobordisms of tangles

**Definition (Cobordisms)**

Let $B$ be a finite subset of $S^1$. The category $\mathbf{Cob}^3(B)$ consists of

- Objects: closed oriented 1-submanifolds $T$ of the unit disk $D^1$ with $\partial T = B$;
- Morphisms: $C : T \to T'$ is a oriented 2-submanifold of $D^1 \times I$ such that $\partial C = T' \sqcup \bar{T} \sqcup (B \times I)$, where $\bar{T}$ is $T$ with the reverse orientation.

# Graded categories

**Definition**

Let $\mathcal{A}$ be a category. Then an $\mathcal{A}$-graded category is a catgeory $\mathcal{C}$ with a functor $F : \mathcal{C} \to \mathcal{A}$.

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## Definition

A $\mathbb{Z}$-category is a category $\mathcal{C}$ with a map $F : \mathrm{Mor}(C, D) \to \mathbb{Z}$ for all $C, D \in \mathrm{Ob}(\mathcal{C})$ such that for all $C, D, E \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}(C, D)$, $g \in \mathrm{Mor}(D, E)$, we have that

1. $F(g \circ f) = F(g) + F(f)$;
2. $F(\mathrm{id}_C) = 0$.

# Shift of graded categories

- If $\mathcal{C}$ is a $\mathbb{Z}$-graded category, then we can shift objects of $\mathcal{C}$ by $m \in \mathbb{Z}$.
  In that case, we have
  $\text{Mor}(\mathcal{O}_1\{m_1\}, \mathcal{O}_2\{m_2\}) = \text{Mor}(\mathcal{O}_1, \mathcal{O}_2)$, but if
  $f \in \text{Mor}(\mathcal{O}_1, \mathcal{O}_2)$ has $F(f) = d$, then
  $f \in \text{Mor}(\mathcal{O}_1\{m_1\}, \mathcal{O}_2\{m_2\})$ has $F(f) = d + m_2 - m_1$.

# Graded categories?

- If $\mathcal{C}$ is a $\mathbb{Z}$-graded category, then we say that $F(f) = d \in \mathbb{Z}$ for a morphism $f$ in $\mathrm{Mat}(\mathcal{C})$ if $F(f_{ij}) = d$ for all entries.
- A similar statement holds for $\mathrm{Kom}(\mathcal{C})$ (or $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}))$).

# The degree of a cobordism (part 1)

We pick $F = \deg$.

**Definition**

Let $C$ be a morphism of $\mathbf{Cob}^3(B)$. Then the degree of $C$ is given by $\deg(C) = \chi(C) - \frac{1}{2}|B|$, where $\chi(C)$ is the Euler characteristic of $C$. and where $|B|$ is the number of vertical boundary components of $C$.

**Example**

1. deg() = deg() $= 1$;
2. deg() $= -1$.

# The degree of a cobordism (part 2)

## Proposition

The degree of a cobordism is additive under compositions.

## Proof.

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**Proof.**

Let $C, D$ be composable morphisms of $\mathbf{Cob}^3(B)$.
Claim: $\chi(D \cap C) = \frac{1}{2}|B|$.

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## Proof.

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Claim: $\chi(D \cap C) = \frac{1}{2}|B|$.
Given the claim, we find that

$$\begin{aligned}\deg(D \circ C) &= \deg(D \cup C) = \chi(D \cup C) - \frac{1}{2}|B| \\ &= \chi(D) + \chi(C) - \chi(D \cap C) - \frac{1}{2}|B| \\ &= \chi(D) + \chi(C) - \frac{1}{2}|B| - \frac{1}{2}|B| = \deg(D) + \deg(C).\end{aligned}$$

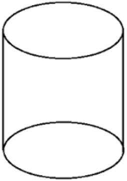
# The degree of a cobordism (part 3)

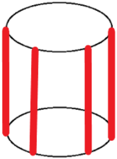
## Proposition

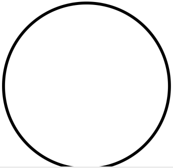
The degree of a cobordism is additive under planar diagrams.

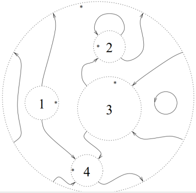
## Lemma

*The Euler characteristic of a cylinder is 0.*
*The Euler characteristic of two circles connected by $\ell$ lines is $-\ell$.*
*The Euler characteristic of a disk is 1.*









# The degree of a cobordism (part 4)

**Proposition**

The degree of a cobordism is additive under planar diagrams.

**Proof.**

Let $D$ be a planar diagram with two inputs, $|B_D|$ points on the outer circle, and $|B_i|$ on the $i$-th input. Furthermore, let $C_i$ be a morphism of $\mathbf{Cob}^3(B_i)$.

<u>Claim:</u> $\chi(D) = \frac{1}{2}(|B_D| + |B_1| + |B_2|)$.

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# The degree of a cobordism (part 5)

**Proof.**

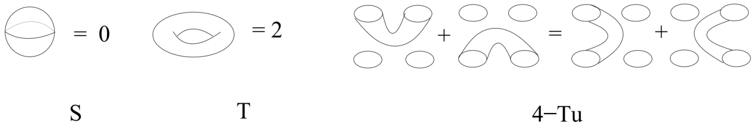
Given the claim, we find that

$$\begin{aligned}\deg(D(C_1, C_2)) &= \chi((D(C_1, C_2))) - \frac{1}{2}|B_D| \\ &= \chi(D \cup C_1 \cup C_2) - \frac{1}{2}|B_D| \\ &= \chi(D) + \chi(C_1) + \chi(C_2) - \chi(D \cap C_1) - \chi(D \cap C_2) - \frac{1}{2}|B_D| \\ &= \frac{1}{2}(|B_D| + |B_1| + |B_2|) + \chi(C_1) + \chi(C_2) - |B_1| - |B_2| - \frac{1}{2}|B_D| \\ &= \chi(C_1) + \chi(C_2) - \frac{1}{2}|B_1| - \frac{1}{2}|B_2| \\ &= \deg(C_1) + \deg(C_2). \quad \square\end{aligned}$$

# Homogeneous relations

## Proposition

The $S$, $T$ and $4Tu$ relations are degree-homogeneous.



# A new complex (part 1)

Let $T$ be a tangle diagram. Consider the complex $\mathrm{Kh}(T)$ given by $\mathrm{Kh}^r(T) := [[T]]^r\{r + n_+ - n_-\}$.

**Theorem**

1. All differentials in $\mathrm{Kh}(T)$ are of degree 0;
2. $\mathrm{Kh}(T)$ is an invariant of $T$ up to degree-0 homotopy equivalences.

# A new complex (part 2)

## Theorem

1. All differentials in $\mathrm{Kh}(T)$ are of degree 0;
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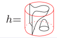
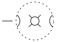
## Proof.

If $f : [[T]]^k \to [[T]]^{k+1}$ has degree $d$, then
$f : \mathrm{Kh}^k(T) \to \mathrm{Kh}^{k+1}(T)$ has degree
$\deg(f) + (k + 1 + n_+ - n_-) - (k + n_+ - n_-) = \deg(f) + 1.$

1. deg(  ) $= -1$.
2. deg(  ) = deg(  ) = deg(  ) $= 1$.

# How to find an invariant

A functor $\mathcal{F} : \text{Cob}^3_{/\ell}(B) \to \textbf{Vect}_{\mathbb{Q}}$

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If we change $\mathbf{Vect}_{\mathbb{Q}}$ into $\mathbf{grVect}_{\mathbb{Q}}$, and if $\mathcal{F}$ respects degrees, (i.e. $\deg(\mathcal{F}(f)) = \deg(f)$), then $H(\mathcal{F}\mathrm{Kh}(T))$ is a graded invariant of $T$.

# Our Frobenius algebra

Our choice of Frobenius algebra is $V = \mathbb{Q}[x] \oplus \mathbb{Q}[1]$, $\deg(x) = -1$, $\deg(1) = 1$

- $m(1 \otimes 1) = 1$, $m(x \otimes 1) = x = m(1 \otimes x)$, $m(x \otimes x) = 0$;
- $\Delta(1) = 1 \otimes x + x \otimes 1$, $\Delta(x) = x \otimes x$;
- $\eta(1) = 1$;
- $\epsilon(1) = 0$, $\epsilon(x) = 1$.

Call the corresponding TQFT $\mathcal{F}$.

# Properties of $\mathcal{F}$ (part 1)

## Proposition

$\mathcal{F}$ respects degrees and descends to a functor $\text{Cob}^3_{/\ell}(\emptyset) \to \textbf{grVect}_{\mathbb{Q}}$.

## Proof.

Check the degrees! What is the degree of a pair of pants?

$\square$

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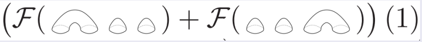
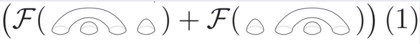
(*S*) $\epsilon \circ \eta = 0$, as $\epsilon(\eta(1)) = \epsilon(1) = 0$.

(*T*) The torus $T$ is given by $\epsilon \circ m \circ \delta \circ \eta$. We have that

$$\begin{aligned} \epsilon(m(\Delta(\eta(1)))) &= \epsilon(m(\Delta(1))) = \epsilon(m(1 \otimes x + x \otimes 1)) \\ &= \epsilon(x + x) = 2. \end{aligned} \tag{4}$$

# Properties of $\mathcal{F}$ (part 2)

**Proof.**

(4*Tu*) We prove that $L :=$  is equal to $R :=$ .

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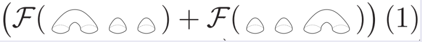
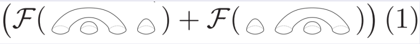
(4*Tu*) We prove that $L :=$ 

is equal to $R :=$ 

.
We find that

$$\begin{aligned} L :&= \Delta\eta(1) \otimes \eta(1) \otimes \eta(1) + \eta(1) \otimes \eta(1) \otimes \Delta\eta(1) \\ &= 1 \otimes x \otimes 1 \otimes 1 + x \otimes 1 \otimes 1 \otimes 1 \\ &\quad + 1 \otimes 1 \otimes 1 \otimes x + 1 \otimes 1 \otimes x \otimes 1. \end{aligned} \tag{5}$$

By applying permutations, this is equal to $R$.