Khovanov homology à la Bar-Natan

Given a set S, the Booolean lattice of S is to poset $\mathbb{B}(S)$ of subsets ordered by inclusion.

$S = \{1, 2, 3\}$

$$\begin{tikzcd}
 & 1 \arrow[r] \arrow[dr] & 12 \arrow[dr] & \\
\emptyset \arrow[ur] \arrow[r] \arrow[dr] & 2 \arrow[ur] \arrow[dr] & 13 \arrow[r] & 123 \\
 & 3 \arrow[ur] \arrow[r] & 23 \arrow[ur] &
\end{tikzcd}$$

Any poset $(\mathbb{P}, \leq)$ gives rise to a category

$$\mathbb{P} \left\{ \begin{array}{l} \underline{\text{obj}}: \text{elmt's of } \mathbb{P} \\ \underline{\text{arrows}}: \exists!\ a \to b \text{ if } a \leq b \end{array} \right.$$

If $S = cr(D) \equiv$ crossings of a link diag D, then our constr. gives rise to a functor

$$F_D : \mathbb{B}(cr(D)) \longrightarrow grVect_{\mathbb{Q}}$$

Our construction is a 2-step process:

1. – Make resolutions (geom. objects)
2. – Associate alg data (applying TQFT)

$$\begin{tikzcd}
\mathcal{B}(cr(D)) \arrow[r] \arrow[rr, bend right, "F_D"'] & Cob_2 \arrow[r, "TQFT"] & grVect_{\mathbb{Q}}
\end{tikzcd}$$

<u>BN's idea</u>:

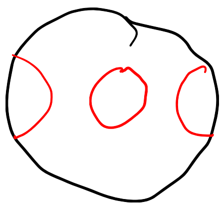
delay this step

<u>Goal</u>: Define $C^*(D)$ without applying TQFT. So we need

Ⓐ Take $\oplus$

Ⓑ Take maps between $\oplus$'s.

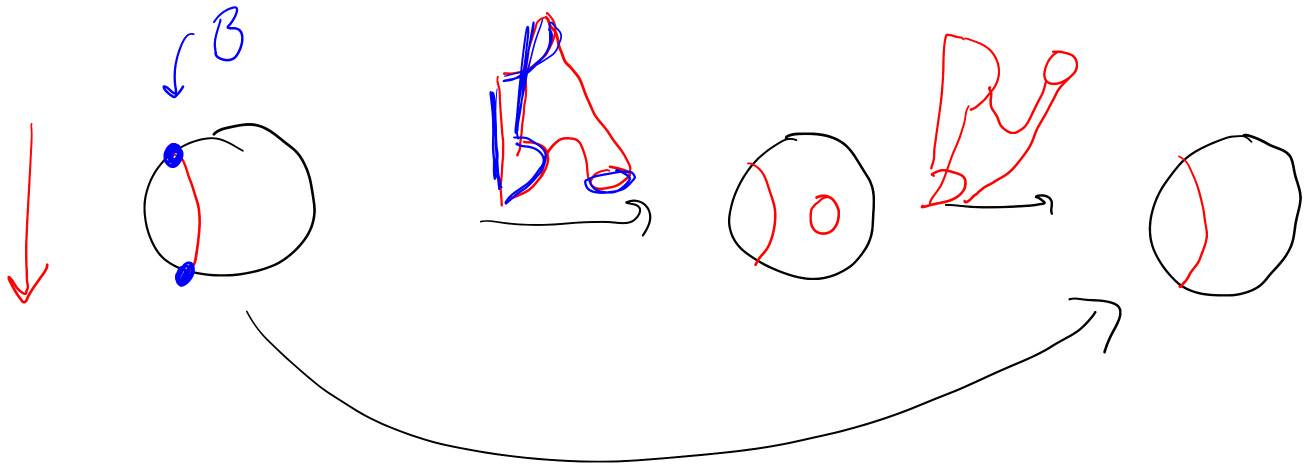
BN also extends the theory to tangles ($\coprod_K I \hookrightarrow D^2 \times [-1,1]$ st $\partial T \subset D^2 \times \{0\}$)



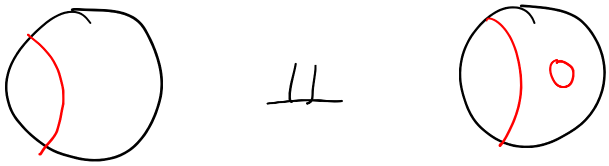
Replace Cob_2 by Cob_2^B (where $B = \partial T$) (B a finite set)

$$Cob_2^B = \begin{cases} \underline{\text{obj}}: \text{ compact oriented } 1\text{-mnfds } L \text{ st } \partial L = B \\ \underline{\text{arrows}}: \text{ boundary-preserving, orientation pres. homeomorph} \\ \text{classes of bordisms } M: L \to L' \text{ st} \\ \qquad \partial M = -L \sqcup L' \sqcup (B \times I) \end{cases}$$

Eg:



⚠ Cob_2^B is <u>not</u> monoidal (at least in the obvious way)



Def: Let $\mathscr{C}$ be a category

1) $\mathscr{C}$ is <u>Ab-category</u> if $\mathrm{Hom}_{\mathscr{C}}(A,B)$ is a abelian gps & the composite $\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$ is a $\mathbb{Z}$-bilinear map.

2) $\mathscr{C}$ is <u>additive</u> if it is Ab-category, it has a zero-object (terminal & initial object) & it has finite coproducts ($\Rightarrow$ it has <u>products = coproducts</u>)

(coproduct $= \oplus$, product $= \prod$)

eg: Vector spaces, Ab, Mod_R $\qquad V \times W = V \oplus W$

3) $F : \mathcal{C} \to \mathcal{D}$ between additive categories is <u>additive</u> if there are natural iso's

$$F(0_{\mathcal{C}}) \cong 0_{\mathcal{D}} \quad , \quad F(A \oplus B) \cong F(A) \oplus F(B)$$

<u>Theorem</u>: let $\mathcal{C}$ be category. There is a unique additive category $\overline{\mathcal{C}}$ (the <u>additive closure</u> of $\mathcal{C}$) together w/ a fully faithful embedding st if $\mathcal{D}$ is an additive category and $F : \mathcal{C} \to \mathcal{D}$ then there exists a unique additive functor $\overline{F} : \overline{\mathcal{C}} \to \mathcal{D}$ st

$$\begin{tikzcd} \overline{\mathcal{C}} \arrow[dr, dashed, "\exists ! \overline{F}"] & \\ \mathcal{C} \arrow[u, hook, "i"] \arrow[r, "F"'] & \mathcal{D} \end{tikzcd} \qquad \boxed{\overline{F} i = F}$$

<u>Pf</u>. <u>Uniqueness</u>: Suppose $\widetilde{\mathcal{C}}$ is another such, $j : \mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$

$$\begin{tikzcd} & \overline{\mathcal{C}} \arrow[d, dashed, shift right, "\overline{j}"'] \\ \mathcal{C} \arrow[ur, hook, "i"] \arrow[r, hook, "j"'] & \widetilde{\mathcal{C}} \arrow[u, dashed, shift right, "\overline{i}"'] \end{tikzcd}$$

$\bar{i} \circ \bar{j} = \mathrm{Id}_{\bar{\mathcal{C}}}$

$\left.\begin{array}{l} \bar{j} \circ i = j \\ \bar{i} \circ j = i \end{array}\right\}$

$$\begin{tikzcd} \bar{\mathcal{C}} \arrow[dr, "\bar{i} \circ \bar{j}"] & \\ \mathcal{C} \arrow[u, hook, "i"] \arrow[r, hook, "i"] & \bar{\mathcal{C}} = \mathcal{D} \end{tikzcd}$$

$\bar{i}\bar{j}i = \bar{i} \circ j = i$

$\Rightarrow \boxed{\mathrm{Id} = \bar{i} \circ \bar{j}}$

<u>Existence</u>: $\mathcal{C}$ can be easily turned into a Ab-category by replacing $\mathrm{Hom}_{\mathcal{C}}(A,B)$ by $\mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(A,B)]$

&

$$\begin{array}{ccc} \mathbb{Z}[\mathrm{Hom}(\ ) \times \mathrm{Hom}(\ )] & \longrightarrow & \mathbb{Z}[\mathrm{Hom}(\ )] \\ \cong & \nearrow_{\mathbb{Z}\text{-linear}} & \\ \mathbb{Z}[\mathrm{Hom}(\ )] \otimes \mathbb{Z}[\mathrm{Hom}(\ )] & & \end{array}$$

$\times$ $\mathbb{Z}$-bilinear

$\bar{\mathcal{C}} = \begin{cases} \text{obj: finite formal direct sums } \bigoplus_{k=1}^{n} C_k \text{ (possibly empty)} \\ \text{arrows: } \bigoplus_k C_k \longrightarrow \bigoplus_\ell D_\ell \text{ is a collection (matrix)} \\ \text{of arrows } f_{\ell k} : C_k \longrightarrow D_\ell \end{cases}$

Composite: Modelled by matrix multiplication

$$(g \circ f)_{pq} = \sum_k g_{pk} \circ f_{kq}$$

$$\begin{pmatrix} g_{11} & g_{12} \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} \end{pmatrix}$$

$\overline{\mathcal{C}} = Mat(\mathcal{C})$ in BN paper

Ex: $\overline{\mathcal{C}}$ satisfies the u. property. □

Now consider $\overline{Cob_2^B}$

Def: Let T be a tangle diagram w/ n-crossings. For a resolution α consider the smoothing $T_\alpha \in Cob_2^{\partial T}$. Define

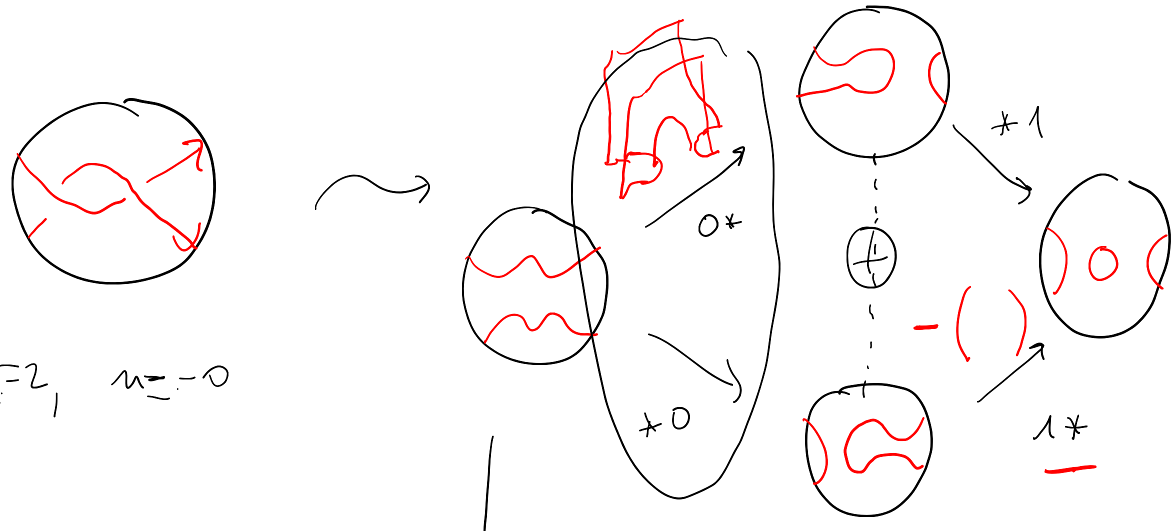
$$C^i(T) := \bigoplus_{\substack{\alpha \in \{0,1\}^n \\ |\alpha| = i + n_-}} T_\alpha \in \overline{Cob_2^{\partial T}}$$

$/\!/$

$[\![T]\!]^i$

in BN paper

Ex :

$T =$ 

$n_+ = 2, \quad n_- = 0$

$$C^*(T) = C^0(T) \xrightarrow{\partial} C^1(T) \xrightarrow{\partial} C^2(T)$$

since $\overline{Cob_2^B}$ is additive,

it has 0 map $\Rightarrow$ exactness makes sense

$\Rightarrow$ can also define $Ch(\overline{Cob_2^B})$

Prop: $\partial^2 = 0$ ie $C^*(T)$

Def: If $\mathcal{C}$ is Ab-category (I have $Ch(\mathcal{C})$)

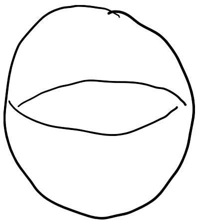
denote $K(\mathcal{C})$ to the quotient category of $Ch(\mathcal{C})$

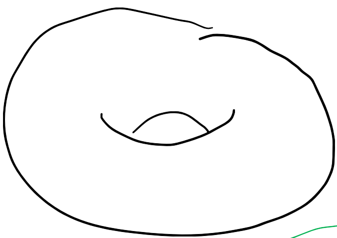
$$\mathrm{Hom}_{K(\mathcal{C})}(A,B) = \mathrm{Hom}_{Ch(\mathcal{C})}(A,B) \big/ \text{chain htpy classes}$$

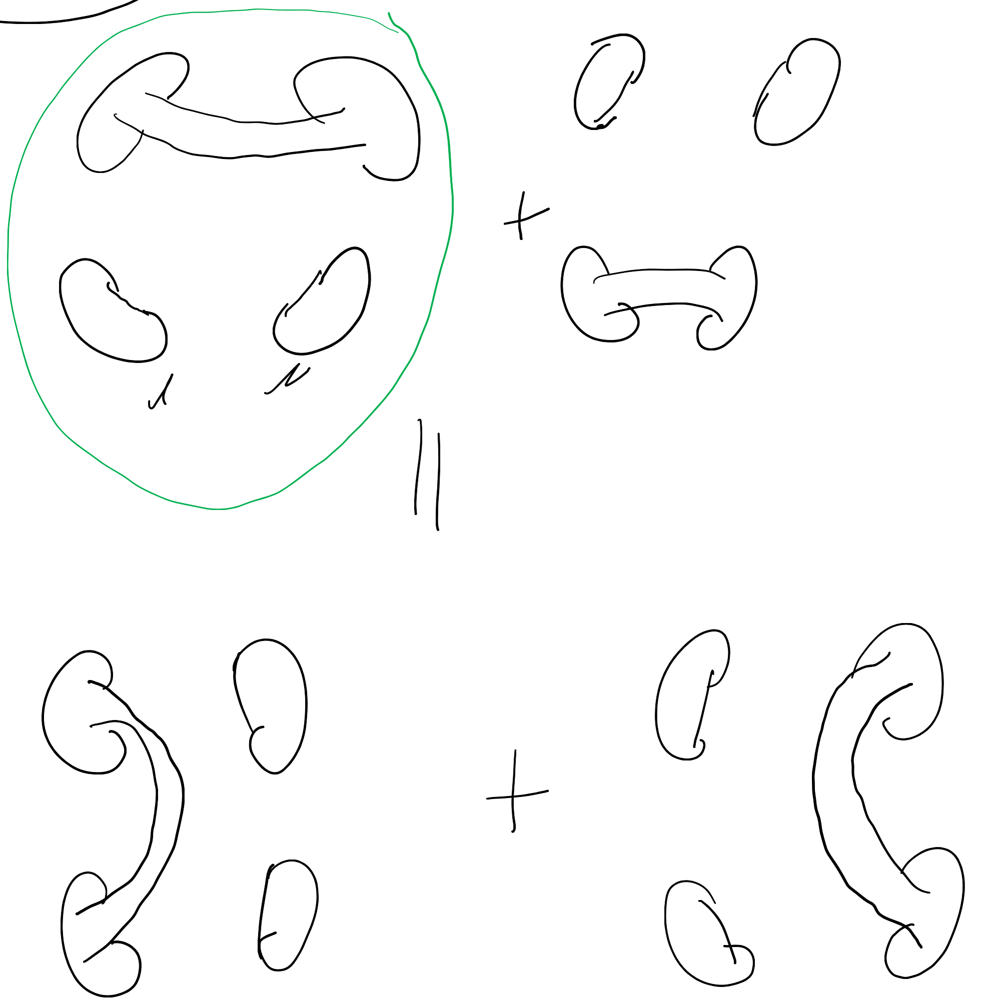
$= Kom(\mathcal{C})$ in BN

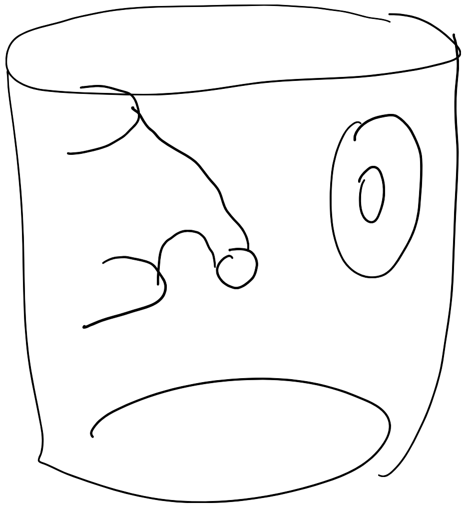
One more ingredient: Quotient Cob_2^B by

$$Cob_2^B/\ell = Cob_2^B \Big/ S, T, 4Tu$$

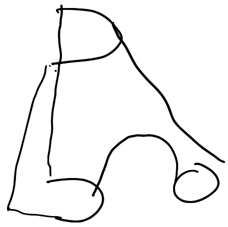
(S)  $= 0$

(T)  $= 2 \cdot$

(4Tu) 



$\overset{(T)}{=} \; 2 \cdot$



<u>Theorem (BN)</u>: The isomorphism class of $C^*(T) \in K\left(\overline{\mathrm{Cob}_2^{\partial T}/\ell}\right)$ is an invariant of T.