Examples & isotopy invariance of Kh

Recall: $$\hat{J}(D) = \sum_{\alpha \in \{0,1\}^n} (-1)^{|\alpha| - n_-} q^{|\alpha| + n_+ - 2n_-} (q + q^{-1})^{k_\alpha}$$

n_+, n_- = pos/neg crossings

α = resolutions

$|\alpha| = \sum$ 1's in α

k_α = # circle components

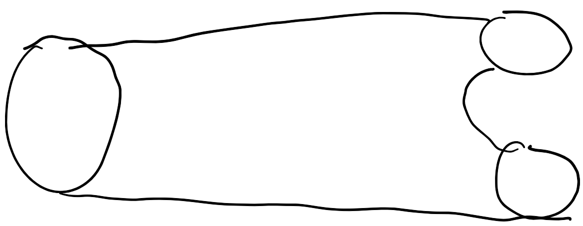
Want: Extend this to a homology theory.

All resolutions form a diagram Γ (T_α's)

there are arrows $\underbrace{T_\alpha \to T_{\alpha'}}$ whenever there is a change $0 \rightsquigarrow 1$.

2-manifold st its boundary is $-T_\alpha \amalg T_{\alpha'}$

$$\|$$



Γ is a diagram in Cob_2

Apply a TQFT to Γ, that is a monoidal functor

$$F : Cob_2 \longrightarrow Vect_{\mathbb{Q}}$$

$$C \longmapsto F(C)$$

$$(C \xrightarrow{f} C') \longmapsto F(f) : F(C) \to F(C')$$

lin map

$$C \amalg C' \longmapsto F(C) \otimes F(C')$$

$$\emptyset \qquad\qquad \mathbb{Q} \quad \text{unit elmt.}$$

$$f, g$$
$$\downarrow$$
$$f \amalg g \longmapsto F(f) \otimes F(g).$$

F is completely determined by $V := F(S^1)$.

$\downarrow$

comm Frob. algebra

Take $V := \dfrac{\mathbb{Q}[x]}{(x^2)} = \mathbb{Q}\, 1_{(1)} \oplus \mathbb{Q}\, x_{(-1)}$

$$\cong H^\bullet(\mathbb{CP}^1; \mathbb{Q})$$

Fact: $H^\bullet(M; k)$ is Frob alg.

closed n-manifold, oriented

$$\left(H_n(M;k) \quad , \quad H^n(M;k) = H_n(M,k)^* \qquad H^\bullet(M,k) = \bigoplus H^n(M;k) \right)$$

k ↑ field

<u>Claim</u>: It is a ring

$1\cdot 1 = 1, \qquad 1\cdot x = x = x\cdot 1, \qquad x\cdot x = 0$

$\Delta(1) = x\otimes 1 + 1\otimes x, \qquad \Delta(x) = x\otimes x$

$\eta(1) = 1, \quad \varepsilon(1) = 0, \qquad \varepsilon(x) = 1.$

Denote F_V to the TQFT det. by V.

$$\underset{\text{in } Cob_2}{\Gamma} \xrightarrow{F_V} \underset{\text{in } Vect_{\mathbb{Q}}}{F_V(\Gamma)}$$

$$F(\Gamma_\alpha) = V^{\otimes k_\alpha}$$

Take $V_\alpha := V^{\otimes k_\alpha}\{|\alpha| + n_+ - 2n_-\}$

And $C^{i,*}(D) := \displaystyle\bigoplus_{\substack{\alpha \\ |\alpha| = i+n_-}} V_\alpha$

so that $\chi_{gr}(C^{**}) = \hat{J}$

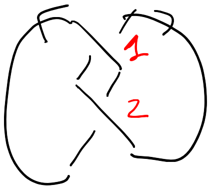
There are differentials $d^i : C^{i,*}(D) \longrightarrow C^{i+1,*}(D)$

$$d^i(v) = \sum_{\substack{\zeta \text{ arrows st} \\ \mathrm{Tail}(\zeta)=\alpha}} \mathrm{sign}(\zeta)\cdot d_\zeta(v)$$

$v \in V_\alpha \subset C^{i,*}(D)$

$\mathrm{sign}(\zeta) = (-1)^{\#(1\text{'s before } *)}$

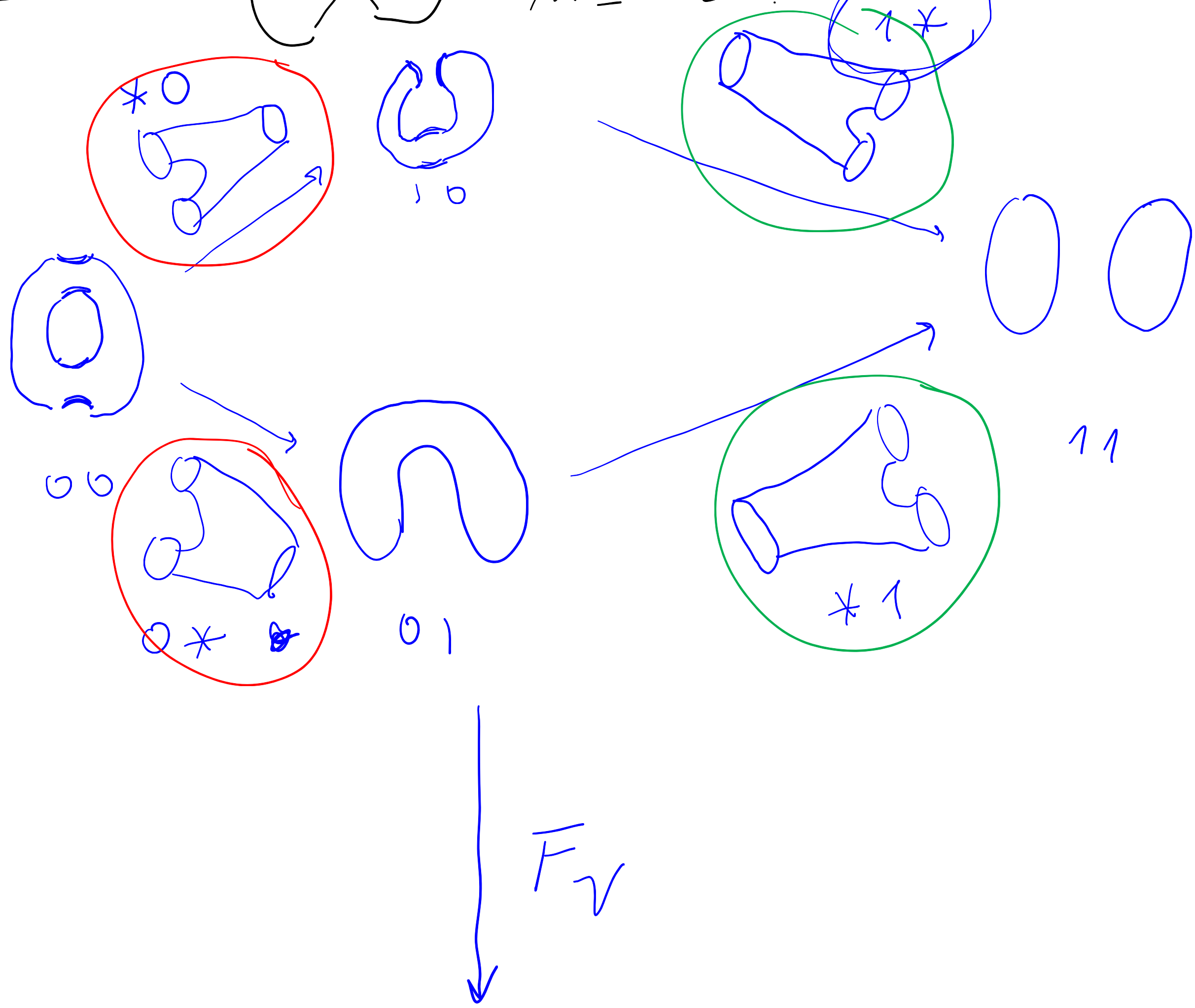
<u>Hopf link</u> :



$n_+ = 0$

$n_- = 2$

$\Gamma =$



$$\begin{tikzcd} & V\{-3\} \arrow[dr, "-\Delta"] \arrow[d, phantom, "\oplus"] & \\ V\otimes V\{-4\} \arrow[ur, "+m"] \arrow[dr, "+m"'] & & V\otimes V\{-2\} \\ & V\{-3\} \arrow[ur, "\pm\Delta"'] & \end{tikzcd}$$

$1*$ $*1$

$$\begin{tikzcd} C^{-2,*}(D) \arrow[r, "{(m,m)}"] & C^{-1,*}(D) \arrow[r, "-\Delta\oplus\Delta"] & C^{0,*}(D) \end{tikzcd}$$

$$Kh^{-2,*}(D) = Ker\left(V\otimes V\{-4\} \xrightarrow{(m,m)} V\{-3\}\oplus V\{-3\}\right)$$

$$\begin{aligned} 1\otimes 1 &\longmapsto (1,1) \\ x\otimes 1 &\longmapsto (x,x) \\ 1\otimes x &\longmapsto (x,x) \\ x\otimes x &\longmapsto (0,0) \end{aligned}$$

$$Ker = \langle \underbrace{x\otimes x}_{-2-4=-6},\ \underbrace{x\otimes 1 - 1\otimes x}_{-4} \rangle$$

<u>Upshot</u>: $Kh^{-2,-6}(D) = \mathbb{Q}$

$Kh^{-2,-4}(D) = \mathbb{Q}$

q-deg \ hom-deg	−2	0
0		$\mathbb{Q}$
−2		$\mathbb{Q}$
−4	$\mathbb{Q}$	
−6	$\mathbb{Q}$	