Overview of $\widehat{HFK}$ and Kh

Q (Rasmussen, 05): What is a Knot homology theory?

(To be answered at the end)

Q: What is homology? If X is a finite CW-complex (eg a finite simplicial complex) and $c_i = \#\, i$-cells (i-simplices) then the Euler characteristic of X is $\chi(X) := c_0 - c_1 + c_2 - \dots$.

Euler already showed that this is an invariant for polyhedra ($V-E+F=2$) but why is it an invariant for spaces, ie, why does it not depend on the CW/simplicial structure we consider in X?

If X is an space, and k is a field (eg $k=\mathbb{R}$ or $\mathbb{Z}/2$), there is a graded vector space $H(X;k) = \bigoplus_{n\geq 0} H_n(X;k)$ called the singular homology of X with the property that

$$\chi(X) = \chi(H(X;k)) = \sum_{n\geq 0} (-1)^n \dim H_n(X;k) .$$

$$\begin{tikzcd} \text{Cat number 1} & \boxed{H(X;k)} \arrow[d, bend left, "\chi"] \\ \text{Cat. number 0} & \boxed{\chi(X)} \end{tikzcd}$$

$H(X)$ carries much more topological information about X.

Properties:

1) $H : Top \longrightarrow grVect_k$ is a functor. This means

   - For a continuous map $f : X \to Y$ there is a map $f_* : H(X) \to H(Y)$

   (in particular, if $X \cong Y \Rightarrow H(X) \cong H(Y)$)

2) $H(X)$ only depends on the homotopy type of X

3) $H(*) \cong k$ concentrated in degree 0.

4) $H(X \times Y) \cong H(X) \otimes H(Y) \quad \left(:= \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \right)$

5) If $A \subset X$ is a nice* subspace, then there is an exact triangle

$$\begin{tikzcd} H(A) \arrow[rr] & & H(X) \arrow[dl] \\ & H(X/A) \arrow[ul] & \end{tikzcd}$$

$(n > 0)$

*: nice $=$ (X,A) NDR $=$ (X,A) has HEP, A closed

What one would like to do is to mimic this for $\Delta_K(t)$ and $V_K(t)$:

$$\begin{tikzcd}
\text{CN}-1 & \boxed{\widehat{HFK}(K)} \arrow[d, bend left] & & \boxed{Kh(K)} \arrow[d, bend left] \\
\text{CN}-0 & \boxed{\Delta_K(t)} & & \boxed{V_K(t)}
\end{tikzcd}$$

characterised by $\begin{cases} \Delta_{K_+} - \Delta_{K_-} = (t^{1/2} - t^{-1/2})\Delta_{K_0} \\ \Delta_{unknot} = 1 \end{cases}$ $\qquad$ $\begin{cases} q^2 J_{K_+} - q^{2} J_{K_-} = (q - q^{-1}) J_{K_0} \\ J_{unknot} = q + q^{-1} \end{cases}$

A: (Attempt) We should expect a knot homology theory to be a functor

$$H: \text{Links} \longrightarrow \mathcal{A}$$

from some category of links (arrows?) to some abelian category (eg Vect_K or Mod_R)

satisfying

- $H(L)$ only depends on the isotopy type of L
- $H(\text{unknot})$ is specified
- $H(L \amalg L') \cong H(L) \underset{\otimes}{\oplus} H(L')$
- Exact triangles

If these are our expectations then Khovanov homology is bound to please.

Let Links be the category with objects isotopy classes of oriented links in S^3 and arrows $L \to L'$ isotopy classes of link cobordisms, ie, isotopy classes of closed oriented surfaces $\Sigma \subseteq S^3 \times I$ with $\partial \Sigma = -L \sqcup L'$.

<u>Theorem</u>: There is a functor $Kh : \text{Links} \longrightarrow \text{bigrVect}_{\mathbb{Z}/2}$

1) If $\Sigma : L \to L'$ is an isotopy then $Kh(\Sigma) : Kh(L) \xrightarrow{\cong} Kh(L')$ is iso

2) $Kh(\text{unknot}) \cong \mathbb{Z}/2_{(0,1)} \oplus \mathbb{Z}/2_{(0,-1)}$ $\quad \left(Kh(\emptyset) = \mathbb{Z}/2_{(0,0)} \right)$

3) $Kh(L \sqcup L') \cong Kh(L) \otimes Kh(L')$

4) If L is presented by a link diagram a small piece of which is $\times$ then there is a exact triangle

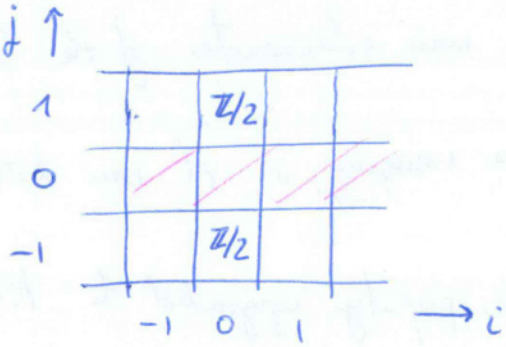
$$\begin{tikzcd}
Kh(\,)(\,) \arrow[rr] & & Kh(\times) \arrow[dl] \\
& Kh(\asymp) \arrow[ul] &
\end{tikzcd}$$

5) The Jones polynomial is the graded Euler characteristic of Kh:

$$\boxed{J_L(q) = \sum_{i,j} (-1)^i \, q^j \, \dim Kh^{ij}(L)}$$

Example:

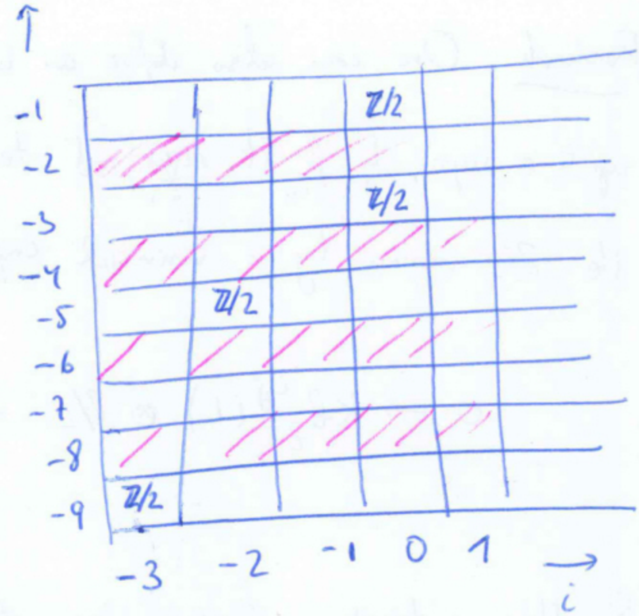
1) Unknot : $Kh^{i,j}(\bigcirc) =$



$$\chi_{gr}(Kh(\bigcirc)) = q^{-1} + q = J_0(q)$$

2) 3_1 left-handed trefoil :

$Kh^{i,j}(3_1) =$



$$\chi_{gr}(Kh(3_1)) = -q^{-9} + q^{-5} + q^{-3} + q^{-1} = J_{3_1}(q).$$

* The even rows are always trivial. This is a more general phenomenon:

Proposition: If $\#L = \text{odd}$, then $Kh^{*,\text{even}}(L) = 0$.

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*

One of the main achievements of the Jones polynomial is that it is sensitive under mirror imaging, ie, it can distinguish K and $\overline{K}$: $J_{\overline{K}}(q) = J_K(q^{-1})$.

How is this property upgraded to Kh?

<u>Proposition</u>: $Kh^{i,j}(\overline{L}) = Kh^{-i,-j}(L)$.

<u>Remark</u>: One can also define an integral Khovanov homology $Kh_{\mathbb{Z}}^{*,*}$ (at least up to a sign, fixing it requires technical modifications). This is related with the $\mathbb{Z}/2$ version by a universal coefficients formula

$$0 \to Kh_{\mathbb{Z}}^{i,j}(L) \otimes \mathbb{Z}/2 \to Kh_{\mathbb{Z}/2}^{i,j}(L) \to \mathrm{Tor}_1^{\mathbb{Z}}\left(Kh_{\mathbb{Z}}^{i+1,j}(L), \mathbb{Z}/2\right) \to 0$$

Something interesting about working with integral Khovanov homology is that one can study torsion (which does not appear over a field). Torsion in $Kh_{\mathbb{Z}}$ is not very well understood.

<u>Downside</u>: Khovanov homology is built out of combinatorial data of a (any) link diagram ("the cube of resolutions"), but so far there is not an intrinsic definition starting with the actual knot in S^3. However topology does come into play with a homotopy-theoretic flavour:

<u>Theorem</u> (Everit–Turner, 2013): For every $n \geq 0$ and a knot diagram D, there is a space $Y_n D$ (built as the holim of some diagram of Eilenberg–MacLane spaces) such that

$$\pi_i(Y_n D) \cong Kh^{i+y(D),\bullet}(D) \qquad , \ 0 \leq i \leq m$$

where $y(D)$ = negative crossings of D and the bullet denotes direct sum over that entry.

<u>Remark</u>. There are some generalisations of Kh: there is a Khovanov–Rozansky $sl(N)$-homology which upgrades the coloured Jones polynomial. The same authors also built a homology theory which upgrades the HOMFLYPT polynomial.

• <u>What about knot Floer homology?</u> (Ozsváth–Szabó, Rasmussen)

Let $K \subset S^3$ be a knot. There are several flavours of the knot Floer homology of K, the simplest being

$$\widehat{HFK}(K) = \bigoplus_{m,s \in \mathbb{Z}} \widehat{HFK}_m(K,s)$$

a bigraded vs over $\mathbb{Z}/2$

This is based on Ozsváth - Szabó 's Heegaard-Floer homology for 3-manifolds:

out of some decomposition $Y = H_1 \cup_\varphi H_2$, where H_i is a regular neighbourhood of $\bigvee_g S^1$ (handlebody) and some extra data, O-S defined a chain complex $\widehat{CF}(Y)$ whose chain homotopy type is an invariant of Y, its Heegaard-Floer homology $\widehat{HF}(Y)$. The differential "counts holomorphic disks".

One defines $\widehat{HFK}(K)$ in a similar fashion. The main achievement is that $\widehat{HFK}$ categorifies the (Conway-normalised) Alexander polynomial:

$$\boxed{\Delta_K(t) = \sum_{m,s} (-1)^m \, t^s \, \dim_{\mathbb{Z}/2} \widehat{HFK}_m(K,s)}$$

Example: 3_1 left-handed trefoil:

$\widehat{HFK}(3_1) =$

$s \uparrow$ \ $m \rightarrow$	0	1	2
2			
1			$\mathbb{Z}/2$
0		$\mathbb{Z}/2$	
−1	$\mathbb{Z}/2$		

$$\chi_g(\widehat{HFK}(3_1)) = t^{-1} - 1 + t = \Delta_{3_1}(t)$$

$\widehat{HFK}$ strengthens some properties of Δ_K:

1) Let $\Delta_K(t) = a_0 + \sum_{s \geq 0} a_s (t^s + t^{-s})$.

Folklore: $g(K) \geq \frac{1}{2}$ breath $\Delta_K(t) = \max \{ s : a_s \neq 0 \}$.

Theorem (Ozsváth-Szabó '04): $\widehat{HFK}$ detects the knot genus:

$$g(K) = \max \{ s : \widehat{HFK}(K, s) \neq 0 \}$$

2) A knot $K \subset S^3$ is fibred if there is a fibre bundle $\Sigma \to S^3 - K \to S^1$ with typical fibre a Seifert surface Σ for K (without boundary)

Folklore: If K is fibered, then $\Delta_K(t)$ is monic, ie $a_{g(K)} = \pm 1$

Theorem (Ghiggini, Ni 07): $\widehat{HFK}$ detects fiberness:

$$K \text{ fibered } \Leftrightarrow \widehat{HFK}(K, g(K)) \cong \mathbb{Z}/2.$$

3) The usual properties of $\Delta_K(t)$ such as $\Delta_K(t) = \Delta_{-K}(t) = \Delta_{\overline{K}}(t)$ or $\Delta_{K \# K'}(t) = \Delta_K(t)\, \Delta_{K'}(t)$ also have their counterpart in $\widehat{HFK}$:

$$\widehat{HFK}(-K) \cong \widehat{HFK}(K)$$

$$\widehat{HFK}(\overline{K}) \cong \left(\widehat{HFK}(K)\right)^*$$

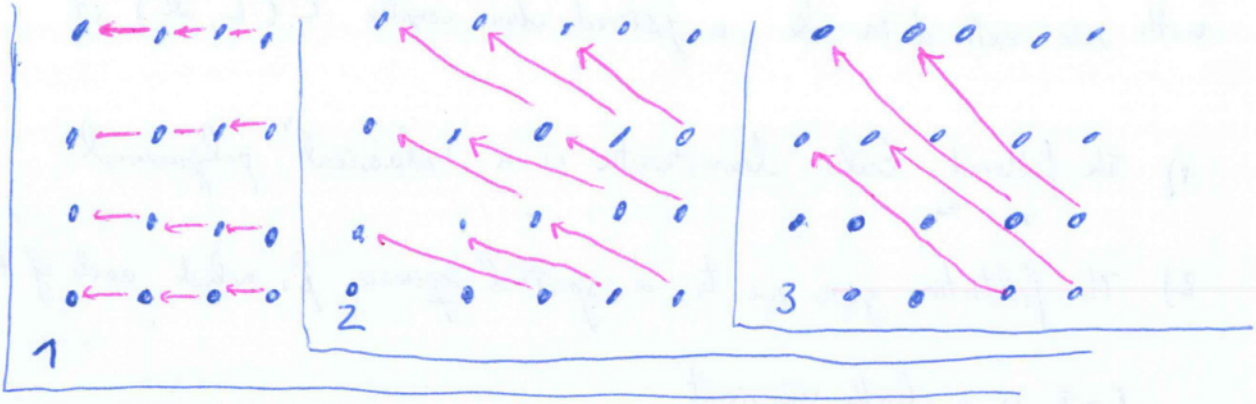
$$\widehat{HFK}(K \# K') \cong \widehat{HFK}(K) \otimes \widehat{HFK}(K').$$

Relation between Kh & $\widehat{HFK}$

Let us finish off with some relations between Kh and $\widehat{HFK}$. To start with, one must say that in both cases the strategy to build these knot homologies is the same: one build a chain complex whose chain homotopy type is a knot invariant.

However, although they share formal properties, the two knot homologies are defined in very different ways: Khovanov is built out of combinatorial data from a knot diagram, whereas the definition of knot Floer has strong topological foundations.

The relation comes usually in the form of a spectral sequence: one can think of a spectral sequence as a book, where every page has a two-dimensional array of abelian groups / vs. On each page there are maps between these groups and they form chain complexes. The homology groups of these complexes are exactly the groups which appear on the next page. The process is finite, and one can typically read off information of the last page from the first one.



<u>Theorem</u> (Ozsváth-Szabó, 05): Let $L \subset S^3$. There is a spectral sequence whose E_2-page is $\widetilde{Kh}(\overline{L})$ (reduced Khovanov homology) converging to $\widehat{HF}(\Sigma(L))$ the Heegaard-Floer homology of the double branched cover of S^3 along L.

<u>Corollary</u>: $|\Delta_K(-1)| = \det L \leq \dim_{\mathbb{Z}/2} \widehat{HF}(\Sigma(L)) \leq \dim_{\mathbb{Z}/2} \widetilde{Kh}(L)$

Recently Kh was related with $\widehat{HFK}$ also through a spectral sequence.

<u>Theorem</u> (Dowlin, 18): Let $K \subset S^3$. There is a spectral sequence with E_2-page $\widetilde{Kh}_{\mathbb{Q}}(K)$ converging to $\widehat{HFK}_{\mathbb{Q}}(\overline{K})$.

<u>Corollary</u> (Dowlin, conjectured by Rasmussen): $\dim \widetilde{Kh}(K) \geq \dim \widehat{HFK}(K)$

Answer to the first question (Rasmussen 05 ; Baldwin - Hedden - Lobb 19) : A

Knot homology theory is a theory, which assigns to an oriented link $L \subset S^3$ together with some extra data $\mathcal{D}$ a filtered chain complex $C(L, \mathcal{D})$ st

1) The filtered Euler characteristic is a "classical" polynomial

2) The filtration gives rise to a spectral sequence for which each of the pages E_i $i \geqslant 2$ is a link invariant

3) $H(C(L, \mathcal{D}))$ depends only on coarse information about L, eg # components, linking number etc.