Isotopy invariance of Kh (take II)

Def: $f_0, f_1 : A \to B$ cochain maps are cochain htpic if there is a seq of maps $P_n : A_n \to B_{n-1}$ st

$$\begin{tikzcd}
A_{n-1} \arrow[r, "(f_0)_{n-1} - (f_1)_{n-1}"] \arrow[d, "d^A"'] & B_{n-1} \arrow[d, "d^B"] \\
A_n \arrow[r, "(\ )\ (\ )"] \arrow[ur, "P_n"] \arrow[d, "d^A"'] & B_n \arrow[d, "d^B"] \\
A_{n+1} \arrow[r, "(\ )-(\ )"] \arrow[ur, "P_{n+1}"] & B_{n+1}
\end{tikzcd}$$

(not comm)

$$f_0 - f_1 = d_B \circ P + P \circ d_A .$$

A map $f : A \to B$ is a chain htpy eq if $\exists\, g : B \to A$ st $gf \simeq \mathrm{Id}_A$, $fg \simeq \mathrm{Id}_B$

$$f : A \xrightarrow{\cong} B \Rightarrow f_* : H(A) \xrightarrow{\cong} H(B).$$

Theorem: Kh is a link invariant.

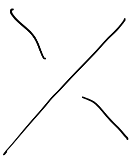
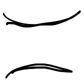
More precisely, any Reid. move $D \rightsquigarrow D'$ induces a chain htpy equivalence $f: C^{*,*}(D') \to C^{*,*}(D)$.

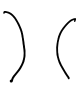
Pf of RI: (RII & RIII in BN's papers)

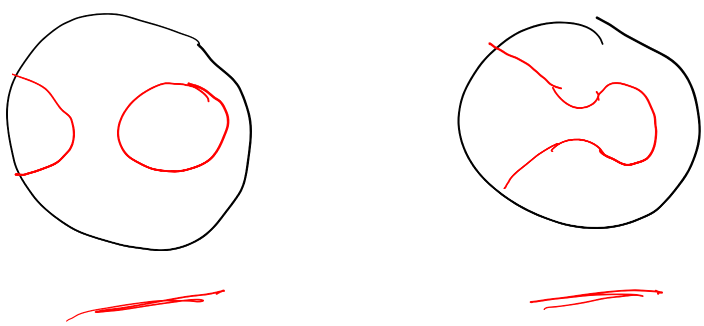
D $\qquad\qquad$ D'

$$C^{i,*}(D) = \bigoplus_{\substack{\alpha \\ |\alpha| = i+n_-}} V_\alpha = \Bigg(\bigoplus_{\substack{\alpha \\ |\alpha| = i+n_- \\ |\alpha'| \\ \alpha = 0\dots}} V_\alpha\Bigg) \oplus \Bigg(\bigoplus_{\substack{|\alpha| = i+n_- \\ |\alpha'|+1 \\ \alpha = 1\dots}} V_\alpha\Bigg)$$

WARNING !!!

 =  1-smooth

 0-smooth



D_0 $\qquad\qquad$ D_1

Claim: $C^{i,*}(D) \cong C^{i,*}(D_0) \oplus C^{i-1,*}(D_1)$

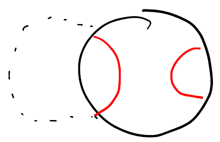
$$\left\{ \begin{array}{c} \text{resolutions} \\ \text{of } D_0 \ (D_1) \end{array} \right\} \overset{\text{bij}}{=\!=} \left\{ \begin{array}{c} \text{resolutions of} \\ D \text{ starting by } 0 \ (1) \end{array} \right\}$$

$\alpha' \longmapsto \alpha = 0\alpha'$ $\quad |\alpha| = |\alpha'|$

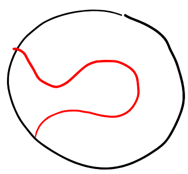
$(\alpha = 1\alpha')$ $\quad |\alpha| = 1 + |\alpha'|$

$$d: C^{i,*}(D) \longrightarrow C^{i+1,*}(D)$$
$$\| $$
$$C_0^{i,*} \oplus C_1^{i-1,*} \longrightarrow C_0^{i+1,*} \oplus C_1^{i,*}$$
$$\| $$
$$\begin{pmatrix} d_0 & \boxed{0} \\ \delta & d_1 \end{pmatrix}$$

δ :

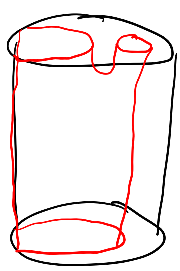


2 - comp



1 - comp

Δ is used for δ at the level of V_α's.

$\delta = F_V\Bigg($  $\Bigg)$

$$f : C^{*,*}(D') \longrightarrow C^{*,*}(D) \cong C^{*,*}(D_0) \oplus C^{*-1,*}(D_1)$$

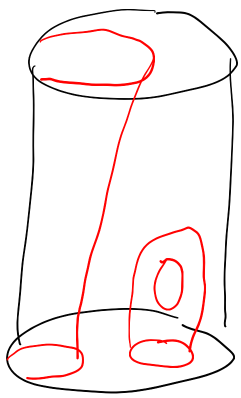
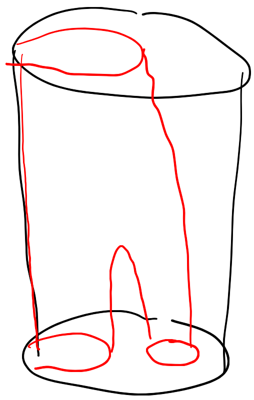
$$f = (f_0, f_1),$$

Choose $f_1 := 0$

f_0: There is another bij $\{\text{resol. of } D_0\} = \{\text{resolutions of } D'\}$

$$\alpha \longleftrightarrow \alpha'$$

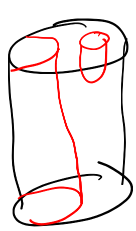
Consider the cobordisms $T_{\alpha'} \longrightarrow T_\alpha$

$F_\nu\Bigg($  $\Bigg) - F_\nu\Bigg($  $\Bigg)$

assemble these maps into f_0.

$$g : C_0^{*,*} \oplus C_1^{*-1} \longrightarrow C^{*,*}(D)$$

$g_0 \oplus g_1$, choose $g_1 := 0$

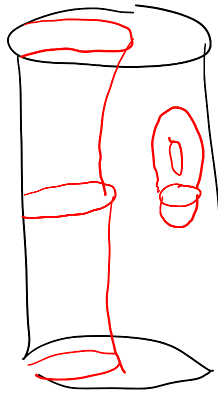
$g_0 := F_\nu\Bigg($  $\Bigg)$

Claim: f is a chain htpy eq w/ homotopy inv g.

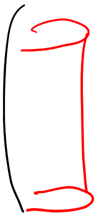
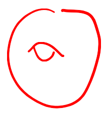
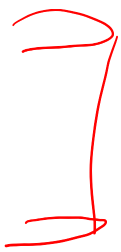
$$g\circ f - \mathrm{Id} = d\circ P + P\circ d \qquad (1)$$

$$f\circ g - \mathrm{Id} = d\circ Q + Q\circ d \qquad (2)$$

(1) Actually $gf = \mathrm{Id}$ (so $P = 0$)

$gf = F_V\Big($  $\Big) - F_V\Big($  $\Big)$

$=$

$F_V\Big($  $\Big) \otimes F\Big($  $\Big) - F_V\Big($  $\Big)$

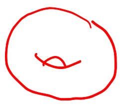
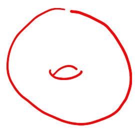
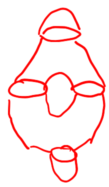
$\underbrace{F\big(\ \cdot\ \big)}_{2\,\cdot}$

$F_V(\ldots) = \mathrm{Id}$, $\quad F_V(\ldots) = \mathrm{Id}$

Ex: For F_V the TQFT det. by $V = \mathbb{Q}[x]/(x^2)$

show as maps $\mathbb{Q} \to \mathbb{Q}$

$F_V\Big($  $\Big) = 0.$

$F_V\Big($  $\Big) = 2.$  $=$ 

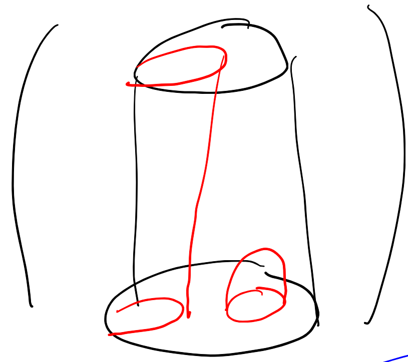
$F_V(\Sigma_g) = 0. \ (g > 1)$

$$= \mathrm{Id}$$

(2) Need to build $Q : C^{*,*}(D) \to C^{*-1,*}(D)$

$$\begin{array}{c} C^{*,*}(D) \\ \| \\ C_0 \oplus C_1 \end{array} \longrightarrow \begin{array}{c} C^{*-1,*}(D) \\ \\ C_0 \oplus C_1 \end{array}$$

$$Q = \begin{pmatrix} 0 & -q \\ 0 & 0 \end{pmatrix}$$

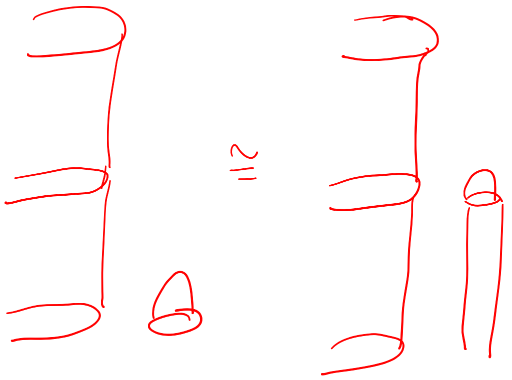
where q is the map F_γ 

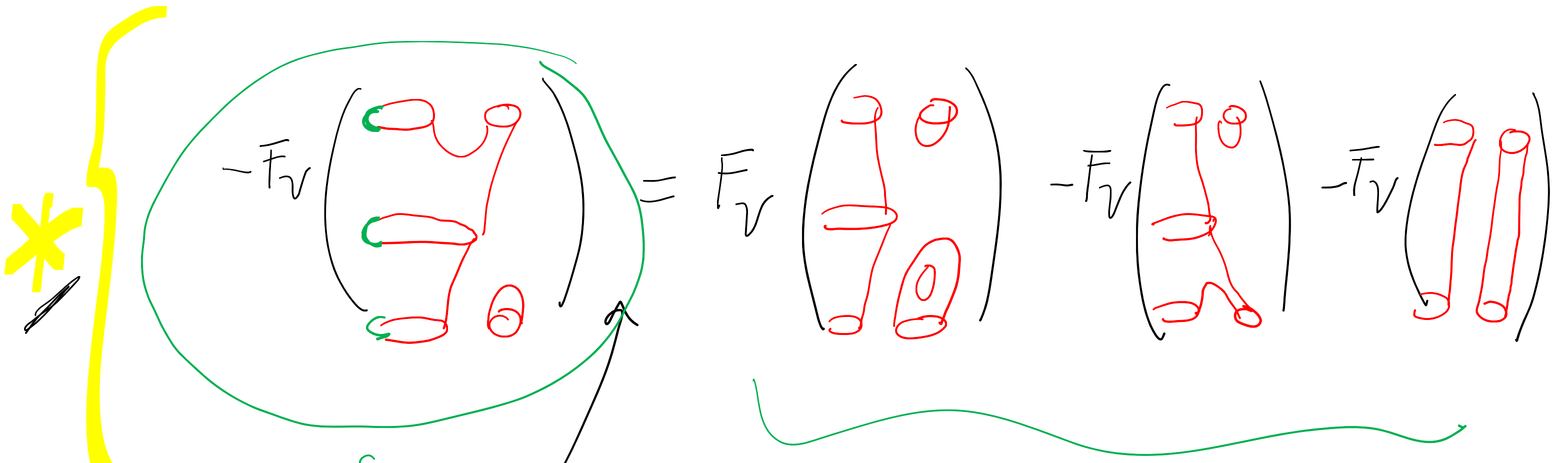
$$fg - \mathrm{Id} = dQ + Qd = \begin{pmatrix} -q\delta & -qd_1 - d_0 q \\ \cancel{0} & -\delta q \end{pmatrix}$$

$$\| $$

$$\begin{pmatrix} f_0 g_0 - \mathrm{Id} & 0 \\ \cancel{0} & -\mathrm{Id} \end{pmatrix}$$

$-F_\gamma$  $\cong -F_\gamma$ 





$-q\delta$ $\qquad$ $f\circ g - \mathrm{Id}$

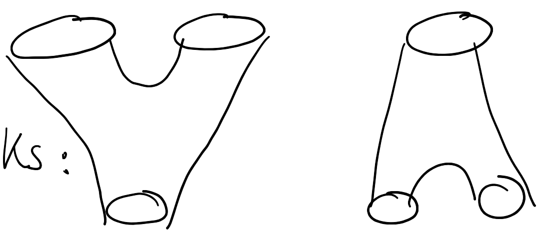
Ex: Check that this is the map $V\otimes V \to V\otimes V$

$a\otimes b \longmapsto -ab\otimes 1$

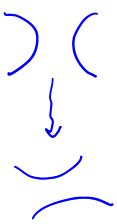
Kh à la BN :

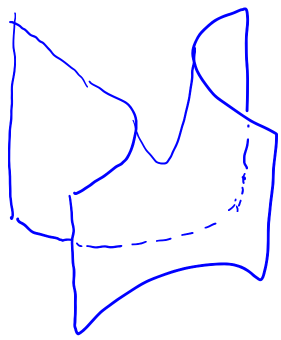
Kh for tangles.

for links:

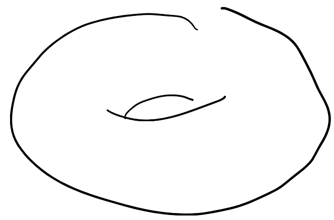


for tangles,

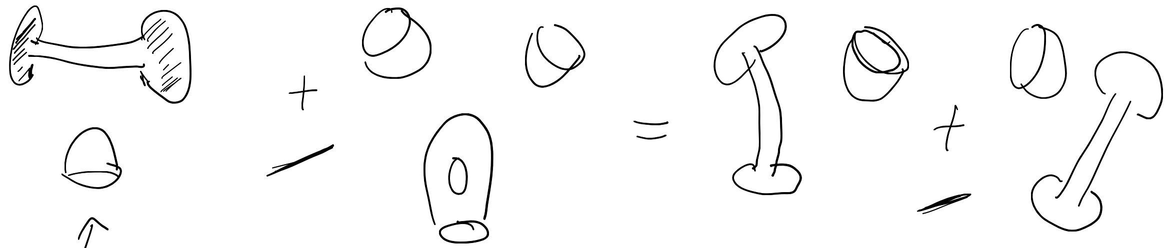




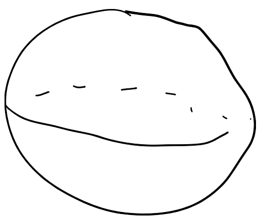
He considers a cat of cobordisms where

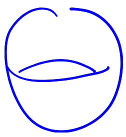
(T)  $= 2.$

(4Tu)



$\exists$ cobordism Σ st $\Sigma \cap B^3$

(S)  $= 0.$ (for RII)

$F_V\left(\right.$  $\left.\right) = \underbrace{(\varepsilon \circ \eta)}_{(\mathbb{Q} \to \mathbb{Q})}(1) = \varepsilon(1) = 0$